On the averaging principle for one-frequency systems. An application to satellite motions.

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Abstract

This paper is related to our previous works [1] [2] on the error estimate of the averaging technique, for systems with one fast angular variable. In the cited references, a general method (of mixed analytical and numerical type) has been introduced to obtain precise, fully quantitative estimates on the averaging error. Here, this procedure is applied to the motion of a satellite in a polar orbit around an oblate planet, retaining only the J_2 term in the multipole expansion of the gravitational potential. To exemplify the method, the averaging errors are estimated for the data corresponding to two Earth satellites; for a very large number of orbits, computation of our estimators is much less expensive than the direct numerical solution of the equations of motion.

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1 Introduction.

Celestial mechanics often requires approximation techniques to compute the motion of bodies over very long times. Giving reliable error estimates on these techniques is a very practical problem, of not simple solution.

In this paper we apply to a typical astronomical problem the general scheme of [1] [2] to estimate the error of the averaging principle, in the case of one fast angular variable ϑ (in the one-dimensional torus) and many slow variables $I = (I^i)$ (the actions). The application we consider refers to the motion of a satellite around an oblate planet: of course, oblateness gives a perturbation of the Keplerian, purely radial gravitational potential. There is a classical way to express the perturbed potential as a series of spherical harmonics, where the multipole moments of the planetary mass distribution appear as coefficients. In the case of axial symmetry, this series only involves zonal harmonics and the ℓ-th term contains a multipole coefficient J_ℓ (for a reminder, see subsection 3E and references therein). For a slightly oblate planet, all these coefficients are small if they are defined in a convenient, dimensionless fashion; often, $J_1 \simeq 0$ and one retains only the $\ell = 2$ term of the previous expansion; the corresponding equations of motion for the satellite are the classical "J_2 problem".

In this paper the J_2 problem is considered in the restricted case of polar orbits; this problem involves a fast angular variable Θ (the angle between the polar axis and the radius vector of the satellite) and three slow variables $\mathtt{I} = (\mathtt{P}, \mathtt{E}, \mathtt{Y})$ representing the parameter, the eccentricity and the argument of the pericenter for the osculating Keplerian ellipse. In principle, all these variables are unknown functions of the "physical" time t; however, it is customary to consider as a "time" variable the angle itself or, rather, the ratio $\mathtt{t}$ of the angle to 2π. With this choice of the time variable, by definition the angle evolves with the law $\Theta(\mathtt{t}) = [2\pi\mathtt{t}]$ (where [ ] indicates equivalence mod.2π); for obvious reasons, $\mathtt{t}$ will be called the *orbit counter*. The evolution of $\mathtt{I} = (\mathtt{P}, \mathtt{E}, \mathtt{Y})$ is described by a set of equations of the form

$$\frac{d\mathtt{I}^i}{d\mathtt{t}} = \varepsilon f^i(\mathtt{I}, [2\pi\mathtt{t}]) \ , \qquad \mathtt{I}^i(0) = I_0^i \ , \tag{1.1}$$

where ε is a small parameter proportional to J_2 (see again subsection 3E). Independently of this specific application, a system of evolution equations like (1.1), with any number of unknown functions $\mathtt{I} = (\mathtt{I}^i)_{i=1,\ldots,d}$ and a small parameter $\varepsilon > 0$, will be called in the sequel a *perturbed periodic system*. To any such system one can associate an averaged system, replacing the functions f^i in (1.1) with their averages over the angle $\Theta = [2\pi\mathtt{t}]$; the solution $\mathtt{J} = (\mathtt{J}^i)$ of the latter is in a function of $\tau := \varepsilon\mathtt{t}$.

The difference between the solutions $\mathtt{I}$ of (1.1) and $\mathtt{J}$ of the averaged system, on a time scale of order $1/\varepsilon$, can be estimated in a fully quantitative way with the general method of [1] [2]; this is what we are going to do in the present work, for

the periodic system describing the polar J_2 problem. As in the cited papers, the expression "fully quantitative" means that our final estimate will have the form

$$|\mathtt{I}^i(\mathfrak{t}) - \mathtt{J}^i(\varepsilon\mathfrak{t})| \leq \varepsilon\mathfrak{n}^i(\varepsilon\mathfrak{t}) \qquad \text{for } \mathfrak{t} \in [0, U/\varepsilon) \ , \tag{1.2}$$

where $\mathfrak{n}^i : [0, U) \to [0, +\infty)$ are computable functions, and U is a constant, whose choice determines quantitatively the interval of observation; we note that U/ε is the total number of orbits.

The availability of an algorithm to construct the estimators $\mathfrak{n}^i$ is the main difference between the approach of [2] and the classical, qualitative result $\mathtt{I}^i(\mathfrak{t}) - \mathtt{J}^i(\varepsilon\mathfrak{t}) = O(\varepsilon)$, see, e.g., [3] [4] [5]. We are aware of the existence of higher order versions of the averaging method, in which the error behaves like $O(\varepsilon^p)$ for $t \in [0, O(1/\varepsilon))$ ($p = 2, 3, ...$; see again [4] [5]). However, the classical theory of higher order averaging gives little more than the qualitative error estimate $O(\varepsilon^p)$; perhaps an extension of our methods could give quantitative estimators of the form $\varepsilon^p\,\mathfrak{n}^i_{(p)}(\varepsilon t)$, but this problem is left to future work.

Let us return to the functions $\mathfrak{n}^i$ in Eq. (1.2). Our method to compute them for the J_2 problem is mainly analytical, but, in its final steps, it requires the numerical solution of an ODE on the interval $[0, U)$; however, this numerical computation is much faster than the direct numerical solution of (1.1) (a fact already appearing in other applications, different from the J_2 problem, considered in our previous papers).

The general treatment proposed in this paper for the polar J_2 problem is subsequently exemplified, choosing for ε and for the initial conditions $I_0 = (P_0, E_0, Y_0)$ the values corresponding to the Earth and to the Polar and Cos-B satellites. In this case, the estimators $\mathfrak{n}^i$ have been computed up to 60000 orbits (i.e., one or two centuries), spending few seconds of CPU time on a PC. To test their reliability, we have computed by direct numerical integration the differences $\mathtt{I}^i(\mathfrak{t}) - \mathtt{J}^i(\varepsilon\mathfrak{t})$; this is a more expensive operation, that we have been able to perform up to 3000 orbits with the same PC; our estimators are quite satisfactory on this interval, since the functions $\mathfrak{t} \mapsto \varepsilon\mathfrak{n}^i(\varepsilon\mathfrak{t})$ are close to the envelopes of the rapidly oscillating functions $|\mathtt{I}^i(\mathfrak{t}) - \mathtt{J}^i(\varepsilon\mathfrak{t})|$.

Of course, the treatment of real satellites of the Earth should include, besides the J_2 gravitational term, other minor perturbations such as: gravitational forces corresponding to higher order moments of the Earth, atmospheric dragging, solar wind, tidal effects due to the Moon gravity. All these effects could be treated, giving rise again to a system of the form (1.1) with more complicated perturbation components εf^i (even for non polar orbits: in general, if the analysis is not restricted to the orbits in a fixed plane, the actions are not three but five). Presumably, the corresponding averaged system and our estimators $\mathfrak{n}^i$ for its error could be computed as well; this is left to future work. In spite of the limitation to the polar J_2 problem, we think that the results of this paper have some interest, because they show that

a general method for error estimates works over very long times on a non-trivial problem.

To conclude, we describe in few words the organization of the paper. In Section 2, we show how to compute the error estimators for the averaging of any periodic system (1.1); this illustration specializes to the periodic case the slightly more general setting of [1] [2] for one-frequency systems. In Section 3, the basic facts on the motion of a satellite in a fixed plane Π are reviewed; the outcomes are the equations of motion for $(\mathtt{I}^i) = (\mathtt{P}, \mathtt{E}, \mathtt{Y})$, for any perturbation of the Keplerian potential keeping the satellite on Π. In Section 4, we specialize the previous equations to the polar J_2 problem; the averaged system is solved, and we construct the algorithm to compute the error estimators $\mathfrak{n}^i$ ($i = {\scriptstyle P, E, Y}$). In Section 5, we perform the computations for the Polar and Cos-B satellites of the Earth. Some details on the constructions of Sections 3-4 are presented in Appendices A-D.

2 Averaging of periodic systems.

2A. Introducing the problem. We consider an open set Λ of (the space of the actions) $\mathbf{R}^d$ and the one-dimensional torus $\mathbf{T}$:

$$\Lambda = \{I = (I^i)_{i=1,\dots,d}\} \subset \mathbf{R}^d \ , \qquad \mathbf{T} := \mathbf{R}/2\pi\mathbf{Z} = \{\vartheta\} \ ; \tag{2.1}$$

the natural projection of the real axis on the torus will be written $[\ ] : \mathbf{R} \to \mathbf{T}$, $x \mapsto [x]$. Now, let

$$f = (f^i)_{i=1,\dots,d} \in C^m(\Lambda \times \mathbf{T}, \mathbf{R}^d), \quad (I, \vartheta) \mapsto f(I, \vartheta) \tag{2.2}$$

(with $m \geqslant 2$); we fix

$$I_0 \in \Lambda \ , \qquad \varepsilon \in (0, +\infty) \ , \tag{2.3}$$

and consider the Cauchy problem

$$\frac{d\mathtt{I}}{d\mathfrak{t}} = \varepsilon f(\mathtt{I}, [2\pi\mathfrak{t}]) \ , \qquad \mathtt{I}(0) = I_0 \ \ : \tag{2.4}$$

the maximal solution (in the future) is a C^{m+1} function $\mathtt{I} : [0, \mathfrak{t}_{\max}) \subset \mathbf{R} \to \Lambda$, $\mathfrak{t} \mapsto \mathtt{I}(\mathfrak{t})$ (with $\mathfrak{t}_{\max} \in (0, +\infty]$). Here and in the sequel, typeface symbols are employed for *functions* of $\mathfrak{t}$ (or $\tau = \varepsilon\mathfrak{t}$)); this allows, for example, to distinguish the above mentioned function $\mathtt{I}$ from a point I of Λ.

The averaged system associated to (2.4) is

$$\frac{d\mathtt{J}}{d\tau} = \overline{f}(\mathtt{J}) \ , \qquad \mathtt{J}(0) = I_0 \ , \tag{2.5}$$

$$\overline{f} = (\overline{f^i})_{i=1,...,d} \in C^m(\Lambda, \mathbf{R}^d) \ , \qquad I \mapsto \overline{f}(I) := \frac{1}{2\pi} \int_{\mathbf{T}} d\vartheta \ f(I, \vartheta) \ ; \tag{2.6}$$

the maximal solution (in the future) is a C^{m+1} function $\mathtt{J} : [0, \tau_{\max}) \subset \mathbf{R} \mapsto \Lambda$, $\tau \mapsto \mathtt{J}(\tau)$. Throughout the paper, we are interested in binding the difference $\mathtt{I}(\mathfrak{t}) - \mathtt{J}(\varepsilon \mathfrak{t})$.

2B. Connections with the framework of [1] [2]. In these papers, we have discussed the Cauchy problem

$$\begin{cases} d\mathtt{I}/d\mathfrak{t} = \varepsilon f(\mathtt{I}, \Theta) \ , & \mathtt{I}(0) = I_0 \\ d\Theta/d\mathfrak{t} = \omega(\mathtt{I}) + \varepsilon g(\mathtt{I}, \Theta) \ , & \Theta(0) = \vartheta_0 \end{cases} \tag{2.7}$$

with f as in (2.2) and $g \in C^m(\Lambda \times \mathbf{T}, \mathbf{R})$, $\omega \in C^m(\Lambda, \mathbf{R})$, $\omega(I) \neq 0$ for all $I \in \Lambda$, $I_0 \in \Lambda, \vartheta_0 \in \mathbf{T}$, $\varepsilon \in (0, +\infty)$, the maximal solution being $(\mathtt{I}, \Theta) : [0, \mathfrak{t}_{\max}) \to \Lambda \times \mathbf{T}$. Precise quantitative estimates have been derived on the difference $\mathtt{I}(\mathfrak{t}) - \mathtt{J}(\varepsilon \mathfrak{t})$, where $\mathtt{J}$ is the solution of (2.5).

Clearly, the perturbed periodic system (2.4) is equivalent to a particular case of (2.7). In fact, let us choose

$$\omega(I) := 2\pi \ , \qquad g(I, \vartheta) := 0 \qquad \text{for all } I \in \Lambda, \, \vartheta \in \mathbf{T} \ ; \qquad \vartheta_0 := 0 \ ; \tag{2.8}$$

then, the equation for Θ in (2.7) is fulfilled with

$$\Theta(\mathfrak{t}) := [2\pi \mathfrak{t}] \tag{2.9}$$

and the equation for $\mathtt{I}$ in (2.7) becomes, with this position, the Cauchy problem (2.4). This remark allows to apply the general results of [1] [2] to the problem (2.4); in the sequel, we report from these papers the minimal elements enabling one to read independently the present paper, i.e.: (i) some basic assumptions required by our approach, and the definitions of certain auxiliary functions; (ii) the final proposition from [2], estimating $\mathtt{I}(\mathfrak{t}) - \mathtt{J}(\varepsilon \mathfrak{t})$ in terms of these auxiliary functions.

2C. A first set of assumptions and auxiliary functions. We use the tensor spaces and the operations on tensors already employed in our previous works; in particular, the spaces $\mathrm{T}^1_1(\mathbf{R}^d)$, $\mathrm{T}^2_0(\mathbf{R}^d)$ and $\mathrm{T}^1_2(\mathbf{R}^d)$ are made, respectively, by the the $(1,1)$, $(2,0)$ and $(1,2)$ tensors over $\mathbf{R}^d$; in the three cases, these are represented as families of real numbers $\mathscr{A} = (\mathscr{A}^i_j)$, $\mathscr{B} = (\mathscr{B}^{ij})$, $\mathscr{C} = (\mathscr{C}^i_{jk})$ $(i, j, k = 1, ..., d)$.

From here to the end of the section the function f, the initial datum I_0 and the parameter ε of Eqs. (2.2) (2.3) are fixed. We stipulate the following:

(1) $s, v \in C^m(\Lambda \times \mathbf{T}, \mathbf{R}^d)$, $p, q, w \in C^{m-1}(\Lambda \times \mathbf{T}, \mathbf{R}^d)$, $u \in C^{m-2}(\Lambda \times \mathbf{T}, \mathbf{R}^d)$ and $\mathscr{M} \in C^{m-2}(\Lambda, \mathrm{T}^1_1(\mathbf{R}^d))$ are the auxiliary functions uniquely defined by the equations

$$f = \overline{f} + 2\pi \, \frac{\partial s}{\partial \vartheta} \ , \quad \overline{s} = 0 \ ; \tag{2.10}$$

$$s = 2\pi \frac{\partial v}{\partial \vartheta} \ , \qquad v(I, 0) = 0 \quad \text{for all } I \in \Lambda; \tag{2.11}$$

$$p := \frac{\partial s}{\partial I} f \ ; \qquad q := \frac{\partial v}{\partial I} f \ ; \tag{2.12}$$

$$p = \overline{p} + 2\pi \frac{\partial w}{\partial \vartheta} \ , \qquad w(I, 0) = 0 \quad \text{for all } I \in \Lambda \ ; \tag{2.13}$$

$$u := \frac{\partial w}{\partial I} f \ ; \qquad \mathscr{M} := \frac{\partial^2 \overline{f}}{\partial I^2} \overline{f} - \left(\frac{\partial \overline{f}}{\partial I} \right)^2 \ . \tag{2.14}$$

The barred symbols $\overline{f}$, $\overline{s}$, $\overline{p}$ denote the averages of f, s, p over the angle, in the sense of (2.6); $\partial/\partial I$ and $\partial^2/\partial I^2$ indicate the Jacobian and the Hessian with respect to the variables $I = (I^i)$. Explicit formulas for s, v and w are

$$s = z - \overline{z} \ , \qquad z(I, \vartheta) := \frac{1}{2\pi} \int_0^{\vartheta} d\vartheta' \ (f(I, \vartheta') - \overline{f}(I)) \ ; \tag{2.15}$$

$$v(I, \vartheta) := \frac{1}{2\pi} \int_0^{\vartheta} d\vartheta' \ s(I, \vartheta') \ ; \qquad w(I, \vartheta) := \frac{1}{2\pi} \int_0^{\vartheta} d\vartheta' \ (p(I, \vartheta') - \overline{p}(I)) \ . \tag{2.16}$$

The above definitions of $s, p, ..., \mathscr{M}$ specialize the general prescriptions of [1] [2] to the case (2.8), of interest in this paper. Similar comments could be added in the sequel, but they will not be repeated.

(2) We introduce the open set

$$\Lambda_{\dagger} := \{ (I, \delta I) \in \Lambda \times \mathbf{R}^d \mid [I, I + \delta I] \subset \Lambda \} \ , \tag{2.17}$$

where $[I, I + \delta I]$ is the closed segment in $\mathbf{R}^d$ with the indicated extremes. From now on, $\mathscr{G} \in C^{m-2}(\Lambda_{\dagger}, \mathrm{T}^1_1(\mathbf{R}^d))$ and $\mathscr{H} \in C^{m-2}(\Lambda_{\dagger}, \mathrm{T}^1_2(\mathbf{R}^d))$ are two functions such that, for all $(I, \delta I) \in \Lambda_{\dagger}$,

$$\overline{p}(I + \delta I) = \overline{p}(I) + \mathscr{G}(I, \delta I) \, \delta I \ , \tag{2.18}$$

$$\overline{f}(I + \delta I) = \overline{f}(I) + \frac{\partial \overline{f}}{\partial I}(I) \, \delta I + \frac{1}{2} \mathscr{H}(I, \delta I) \, \delta I^2, \quad \mathscr{H}^i_{jk}(I, \delta I) = \mathscr{H}^i_{kj}(I, \delta I) \ . \tag{2.19}$$

$\mathscr{G}$ and $\mathscr{H}$ are not uniquely determined, if $d > 1$: possible choices, given by Taylor's formula, are

$$\mathscr{G}(I, \delta I) := \int_0^1 dx \, \frac{\partial \overline{p}}{\partial I}(I + x \delta I) \ , \quad \mathscr{H}(I, \delta I) := 2 \int_0^1 dx \, (1 - x) \frac{\partial^2 \overline{f}}{\partial I^2}(I + x \delta I) \ . \tag{2.20}$$

(3) From now on, $[0, U)$ $(U \in (0, +\infty])$ is a fixed interval where the solution $\mathtt{J}$ of the averaged system (2.5) is assumed to exist. We denote with $\mathtt{R} : [0, U) \to \mathrm{T}^1_1(\mathbf{R}^d)$, $\tau \mapsto \mathtt{R}(\tau)$ and $\mathtt{K} : [0, U) \to \mathbf{R}^d$, $\tau \mapsto \mathtt{K}(\tau)$ the solutions of the Cauchy problems

$$\frac{d\mathtt{R}}{d\tau} = \frac{\partial \overline{f}}{\partial I}(\mathtt{J}) \, \mathtt{R} \ , \qquad \mathtt{R}(0) = 1_d \ ; \tag{2.21}$$

$$\frac{d\mathtt{K}}{d\tau} = \frac{\partial \overline{f}}{\partial I}(\mathtt{J})\,\mathtt{K} + \overline{p}(\mathtt{J}) \ , \qquad \mathtt{K}(0) = 0 \ ; \tag{2.22}$$

these are C^m, and exist on the whole interval $[0, U)$ due to the linearity of the above differential equations. By standard arguments, one proves that $\mathtt{R}(\tau)$ is an invertible matrix for all $\tau \in [0, U)$, and derives for $\mathtt{K}$ the explicit formula $\mathtt{K}(\tau) = \mathtt{R}(\tau) \int_0^\tau d\tau' \, \mathtt{R}(\tau')^{-1} \overline{p}(\mathtt{J}(\tau'))$.

2D. Some more auxiliary functions. We maintain the assumptions and notations of the previous items (1) (2) (3). In [2] we have introduced a second set of auxiliary functions, related to $s, p, \ldots.\mathtt{R}, \mathtt{K}$ and to any separating system of seminorms on $\mathbf{R}^d$. For simplicity, here we consider the seminorms giving the absolute values of each component of vectors and tensors in $\mathbf{R}^d$; what follows is an adaptation of [2] to this choice. From now on, i, j, k range in $\{1, ..., d\}$ and we use Einstein's summation convention on repeated indices.

(4) For $J \in \mathbf{R}^d$ and $\varrho = (\varrho^i) \in [0, +\infty]^d$, we put

$$B(J, \varrho) := \{ I \in \mathbf{R}^d \mid |I^i - J^i| < \varrho^i \ \forall \ i \} \ . \tag{2.23}$$

(5) $\rho = (\rho^i) \in C([0, U), [0, +\infty]^d)$ is a function such that

$$B(\mathtt{J}(\tau), \rho(\tau)) \subset \Lambda \qquad \text{for } \tau \in [0, U) \ . \tag{2.24}$$

We put

$$\Gamma_\rho := \{ (\tau, r) \ \in [0, U) \times [0, +\infty)^d \mid r^i < \rho^i(\tau) \ \forall \ i \ \} \ . \tag{2.25}$$

(6) a^i, b^i in $C^2(\Gamma_\rho, [0, +\infty))$ and $c^i, d^i_j, e^i_{jk} = e^i_{kj} \in C^1(\Gamma_\rho, [0, +\infty))$ are functions such that for any $\tau \in [0, U)$, $\delta J \in B(0, \rho(\tau))$, $\vartheta \in \mathbf{T}$,

$$\left| \Big(s(\mathtt{J}(\tau) + \delta J, \vartheta) - \mathtt{R}(\tau) s(I_0, \vartheta_0) - \mathtt{K}(\tau) \Big)^i \right| \leqslant a^i(\tau, |\delta J|) \ , \tag{2.26}$$

$$\left| \Big(w(\mathtt{J}(\tau) + \delta J, \vartheta) - \frac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau))\, v(\mathtt{J}(\tau) + \delta J, \vartheta) \Big)^i \right| \ \leqslant b^i(\tau, |\delta J|) \ , \tag{2.27}$$

$$\begin{aligned} &\Big| \Big(u(\mathtt{J}(\tau) + \delta J, \vartheta) - \frac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau))(w + q)(\mathtt{J}(\tau) + \delta J, \vartheta) \\ &- \mathscr{M}(\mathtt{J}(\tau)) v(\mathtt{J}(\tau) + \delta J, \vartheta) \Big)^i \Big| \leqslant c^i(\tau, |\delta J|) \ , \end{aligned} \tag{2.28}$$

$$|\mathscr{G}^i_j(\mathtt{J}(\tau), \delta J)| \leqslant d^i_j(\tau, |\delta J|) \ , \tag{2.29}$$

$$|\mathscr{H}^i_{jk}(\mathtt{J}(\tau), \delta J)| \leqslant e^i_{jk}(\tau, |\delta J|) \ . \tag{2.30}$$

In the above, one always intends

$$|\delta J| := (|\delta J|^i)_{i=1,\ldots,d} \ . \tag{2.31}$$

The functions c^i, d^i_j, e^i_{jk} are assumed to be non-decreasing with respect to the variable r, i.e.,

$$(\tau, r), (\tau, r') \in \Gamma_\rho, \quad r^j \leqslant r'^j \ \forall\, j \quad \Rightarrow \quad c^i(\tau, r) \leqslant c^i(\tau, r') \tag{2.32}$$

and similarly for d^i_j, and e^i_{jk}.
(7) Given $a^i, ..., e^i_{jk}$, we define the functions

$$\alpha^i \in C^2(\Gamma_\rho, [0, +\infty)), \quad \alpha^i(\tau, r) := a^i(\tau, r) + \varepsilon b^i(\tau, r) \ , \tag{2.33}$$

$$\gamma^i \in C^1(\Gamma_\rho \times [0, +\infty)^d, [0, +\infty)), \tag{2.34}$$

$$\gamma^i(\tau, r, \ell) := c^i(\tau, r) + d^i_j(\tau, r)\ell^j + \frac{1}{2} e^i_{jk}(\tau, r)\ell^j \ell^k \ .$$

(8) $R^i_j \in C^1([0, U), [0, +\infty))$ and $P^i_j \in C([0, U), [0, +\infty))$ are functions such that, for $\tau \in [0, U)$,

$$|\mathtt{R}^i_j(\tau)| \leqslant R^i_j(\tau) \ , \quad |(\mathtt{R}^{-1})^i_j(\tau)| \leqslant P^i_j(\tau) \ . \tag{2.35}$$

2E. The main result on general periodic systems. In [2], we have shown how to obtain estimators for $\mathtt{I}(\mathtt{t}) - \mathtt{J}(\varepsilon \mathtt{t})$ solving a system of integral equations, or of equivalent differential equations. Here we only report the final differential reformulation, that will be used in the present paper. Keeping in mind the previous items (1)–(8), we have the following statement.

2.1 *Proposition.* *(i) Assume there are $\ell_* = (\ell^i_*) \in [0 + \infty)^d$, $(A^i_j) \in [0, +\infty)^{d^2}$ and $\sigma = (\sigma^i) \in (0 + \infty)^d$, such that*

$$\Sigma := \Pi_{i=1,...,d}[\ell^i_* - \sigma^i, \ell^i_* + \sigma^i] \subset \Pi_{i=1,...,d}(0, \rho^i(0)/\varepsilon) \tag{2.36}$$

and

$$\mathcal{A} := \max_{i=1,...,d} \sum_{j=1}^{d} A^i_j < 1/\varepsilon \ , \tag{2.37}$$

$$\left| \frac{\partial \alpha^i}{\partial r^j}(0, \varepsilon \ell) \right| \leqslant A^i_j \qquad \textit{for } i, j = 1, ...d, \quad \ell \in \Sigma, \tag{2.38}$$

$$|\alpha^i(0, \varepsilon \ell_*) - \ell^i_*| + \varepsilon A^i_j \sigma^i < \sigma^i \qquad \textit{for } i = 1, ..., d. \tag{2.39}$$

Then, there is a unique $\ell_0 = (\ell^i_0)$ such that

$$\ell_0 \in \Sigma \ , \qquad \alpha(0, \varepsilon \ell_0) = \ell_0 \ . \tag{2.40}$$

(ii) With ℓ_0 as above, let $\mathfrak{m} = (\mathfrak{m}^i), \mathfrak{n} = (\mathfrak{n}^i) \in C^1([0, U), \boldsymbol{R}^d)$ solve the Cauchy problem

$$\frac{d\mathfrak{m}^i}{d\tau} = P^i_j \, \gamma^j(\cdot, \varepsilon \mathfrak{n}, \mathfrak{n}) \ , \qquad \mathfrak{m}^i(0) = 0 \ , \tag{2.41}$$

$$\frac{d\mathfrak{n}^i}{d\tau} = \left(1 - \varepsilon\frac{\partial\alpha}{\partial r}(\cdot,\varepsilon\mathfrak{n})\right)^{-1,i}_{\quad h}\left(\frac{\partial\alpha^h}{\partial\tau}(\cdot,\varepsilon\mathfrak{n}) + \varepsilon R^h_k P^k_j\,\gamma^j(\cdot,\varepsilon\mathfrak{n},\mathfrak{n}) + \varepsilon\frac{dR^h_j}{d\tau}\mathfrak{m}^j\right)\ , \quad (2.42)$$

$$\mathfrak{n}^i(0) = \ell^i_0$$

for all i, with the domain conditions

$$0 < \varepsilon\mathfrak{n}^i < \rho^i\ , \qquad \det\left(1 - \varepsilon\frac{\partial\alpha}{\partial r}(\cdot,\varepsilon\mathfrak{n})\right) > 0 \qquad (2.43)$$

(in the above, $1 - \varepsilon\partial\alpha/\partial r$ stands for the matrix $(\delta^i_j - \varepsilon\partial\alpha^i/\partial r^j)$ $(i,j = 1,...,d)$, and Eq. (2.42) contains the matrix elements of its inverse. We note that (2.41) implies $\mathfrak{m}^i \geqslant 0$). Then, the solution $\mathtt{I}$ of the periodic system (2.4) exists on $[0, U/\varepsilon)$ and

$$|\mathtt{I}^i(\mathsf{t}) - \mathtt{J}^i(\varepsilon\mathsf{t})| \leqslant \varepsilon\mathfrak{n}^i(\varepsilon t) \qquad \textit{for } i = 1,...,d,\ \mathsf{t}\in[0,U/\varepsilon). \qquad (2.44)$$

2.2 *Remarks.* For future use, it is worthy to repeat some considerations of [2].
(a) The previous Proposition mentions ℓ_0, the unique fixed point of the map $\alpha(0,\varepsilon\cdot)$ in the set Σ of Eq. (2.36). The proof given in [2] indicates that $\alpha(0,\varepsilon\cdot):\Sigma\to\Sigma$ has Lipschitz constant $\varepsilon\mathcal{A} < 1$ in the maximum component norm $\|z\| := \max_i |z^i|$, with $\mathcal{A}$ as in (2.37). So, from the standard theory of contractions, we have the iterative construction

$$\ell_0 = \lim_{n\to+\infty} l_n, \qquad l_1 \text{ any point of } \Sigma\ ,\ l_n := \alpha(0,\varepsilon l_{n-1}) \qquad \text{for } n = 2,3,...\ . \quad (2.45)$$

For each $n \geqslant 2$,

$$\|\ell_0 - l_n\| \leqslant (\varepsilon\mathcal{A})^{n-1}\frac{\|l_2 - l_1\|}{(1-\varepsilon\mathcal{A})}\ . \qquad (2.46)$$

One can use the above characterization of ℓ_0 to compute it numerically; in this case one finds the iterates up to a large order n, and then approximates ℓ_0 with l_n.
(b) In typical cases, and also in the forthcoming application to satellites, the Cauchy problem (2.41) (2.42) for the unknown functions $\mathfrak{n}^i, \mathfrak{m}^i$ is solved numerically (by some standard package for ODEs).

2F. A final comment on the functions $a^i, ..., e^i_{jk}$. Let us start with a remark on $b^i, c^i, d^i_j, e^i_{jk}$. These functions are always multiplied by a small factor ε in the main statements on $|\mathtt{I}^i(t) - \mathtt{J}^i(\varepsilon t)|$, i.e., Proposition 2.1; for this reason, in applications it is generally sufficient to determine $b^i, ..., e^i_{jk}$ majorizing roughly the left-hand sides of Eqs. (2.27)–(2.30).

The situation is different for the functions a^i, which are not multiplied by ε; in this case, it is important to determine them estimating accurately the left-hand side of Eq. (2.26) or, at least, the part of order zero in δJ. To this purpose, we note the

existence of a function $\mathscr{S} \in C^{m-1}(\Lambda_\dagger \times \mathbf{T}, \mathrm{T}^1_1(\mathbf{R}^d))$ such that, for all $(I, \delta I) \in \Lambda_\dagger$ and $\vartheta \in \mathbf{T}$,

$$s(I + \delta I, \vartheta) = s(I, \vartheta) + \mathscr{S}(I, \delta I, \vartheta)\delta I \ ; \tag{2.47}$$

a solution of this equation is given by Taylor's formula, i.e.,

$$\mathscr{S}(I, \delta I, \vartheta) := \int_0^1 dx \, \frac{\partial s}{\partial I}(I + x\delta I, \vartheta) \ . \tag{2.48}$$

To go on, we write

$$s(\mathtt{J}(\tau) + \delta J, \vartheta) - \mathtt{R}(\tau)s(I_0, \vartheta_0) - \mathtt{K}(\tau) \tag{2.49}$$

$$= \Big(s(\mathtt{J}(\tau), \vartheta) - \mathtt{R}(\tau)s(I_0, \vartheta_0) - \mathtt{K}(\tau)\Big) + \mathscr{S}(\mathtt{J}(\tau), \delta J, \vartheta)\delta J \ ,$$

which decomposes the function in the left-hand side of (2.26) into a zero order part in δJ plus a reminder controlled by $\mathscr{S}$. Now, assume there are functions $a^i_{(0)} \in C^2([0, U), [0, +\infty))$ and $a^i_j \in C^2(\Gamma_\rho, [0, +\infty))$ such that, for all $\tau \in [0, U)$, $\delta J \in B(0, \rho(\tau))$ and $\vartheta \in \mathbf{T}$,

$$\left| \Big(s(\mathtt{J}(\tau), \vartheta) - \mathtt{R}(\tau)s(I_0, \vartheta_0) - \mathtt{K}(\tau)\Big)^i \right| \leqslant a^i_{(0)}(\tau) \ , \tag{2.50}$$

$$|\mathscr{S}^i_j(\mathtt{J}(\tau), \delta J, \vartheta)| \leqslant a^i_j(\tau, |\delta J|) \ . \tag{2.51}$$

Then, Eq. (2.26) is fulfilled by the function

$$a^i(\tau, r) := a^i_{(0)}(\tau) + a^i_j(\tau, r) r^j \ ; \tag{2.52}$$

of course, this definition must be inserted in Eq. (2.33) for γ. In applications one will determine $a^i_{(0)}$ binding very accurately the left-hand side of Eq. (2.50), and a^i_j binding more roughly the left-hand side of Eq. (2.51); this is convenient, since in Eq. (2.52) for a^i the terms a^i_j are multiplied by r^j and this will finally take the small values $r^j = \varepsilon \mathfrak{n}^j(\tau)$.

The above strategy to determine $a^i, ..., e^i_{jk}$ will be employed in the application of the next sections, concerning satellite motions.

3 Basic facts on satellite motions.

3A. A general setting. A satellite of mass m, position P and velocity $\dot{P}$ is assumed to move around a planet of mass M. As a first approximation, we assume the mass of the planet to be uniformly distributed within a sphere of center O; so,

the gravitational potential energy (per unit mass) of the planet has the Keplerian form

$$V_K(P) = -\frac{GM}{|P-O|} \tag{3.1}$$

where $G \simeq 6.674 \cdot 10^{-11}\,\mathrm{m}^3\mathrm{Kg}^{-1}\mathrm{sec}^{-2}$ is the gravitational constant. The corresponding gravitational force $-m\,\mathrm{grad}V_K(P)$ will be referred to as the Keplerian force, and all the other forces are assumed to be small perturbations of it.

The non-Keplerian force acting on the satellite (of gravitational, or any other nature) will be written as $\varepsilon m \boldsymbol{f}(\dot{P}, P)$, where $\varepsilon > 0$ is a small dimensionless parameter, arising naturally from the analysis of this perturbation; the parameter and the satellite mass are factored out from the perturbation, for convenience. So, the total force acting on the satellite is

$$\boldsymbol{F}(\dot{P}, P) = m\left(-\mathrm{grad}V_K(P) + \varepsilon \boldsymbol{f}(\dot{P}, P)\right) \ . \tag{3.2}$$

We assume the perturbation to keep the satellite on a fixed plane Π passing through O, where we introduce the polar coordinates $\rho(P) := |P-O|$, $\vartheta(P) :=$ angle (in $\mathbf{T}$) between a fixed unit vector $\boldsymbol{k}$ and $P-O$. The equations for the satellite motions in this plane can be written as a first-order system for ρ, ϑ and their derivatives $\dot{\rho}, \dot{\vartheta}$, in the following way:

$$\frac{d\dot{\rho}}{dt} - \rho\dot{\vartheta}^2 + \frac{GM}{\rho^2} = \varepsilon Q_\rho(\dot{\rho}, \dot{\vartheta}, \rho, \vartheta) \ , \qquad \frac{d\rho}{dt} = \dot{\rho} \ , \tag{3.3}$$

$$\rho^2\frac{d\dot{\vartheta}}{dt} + 2\rho\dot{\rho}\dot{\vartheta} = \varepsilon Q_\vartheta(\dot{\rho}, \dot{\vartheta}, \rho, \vartheta) \ , \qquad \frac{d\vartheta}{dt} = \dot{\vartheta} \ , \tag{3.4}$$

where $Q_\rho := \boldsymbol{f}\boldsymbol{\cdot}(\partial P/\partial\rho)$ and $Q_\vartheta = \boldsymbol{f}\boldsymbol{\cdot}(\partial P/\partial\vartheta)$ are the Lagrangian components of the perturbation $\boldsymbol{f}$. In particular, in the conservative case $\boldsymbol{f}(P) = -\mathrm{grad}W(P)$ we have

$$Q_\rho(\rho, \vartheta) = -\frac{\partial W}{\partial \rho}(\rho, \vartheta) \ , \qquad Q_\vartheta(\rho, \vartheta) = -\frac{\partial W}{\partial \vartheta}(\rho, \vartheta) \ . \tag{3.5}$$

3B. Kepler elements. Our aim is to re-express the system (3.3) (3.4) after a change of coordinates $(\dot{\rho}, \dot{\vartheta}, \rho, \vartheta) \mapsto (P, E, Y, \vartheta)$ where P, E, Y are related to the unperturbed Kepler problem (as in Eqs. (3.3) (3.4), with $Q_\rho = 0$ and $Q_\vartheta = 0$) . For the sake of brevity, let us introduce the abbreviations

$$\xi \equiv (\dot{\rho}, \dot{\vartheta}, \rho) \ , \qquad (\xi, \vartheta) \equiv (\dot{\rho}, \dot{\vartheta}, \rho, \vartheta) \ . \tag{3.6}$$

For given (ξ, ϑ), we define

$$\mathsf{o}_{\xi\vartheta} := \text{unperturbed Kepler orbit in } \Pi \text{ issuing from the initial datum } (\xi, \vartheta) \ ; \tag{3.7}$$

as usually, we refer to this as the osculating Kepler orbit. Let

$$\mathscr{D} := \{(\xi, \vartheta) \mid \dot\vartheta > 0 \ , \mathsf{o}_{\xi\vartheta} \text{ is a (nondegenerate) ellipse } \}$$

(circular or rectilinear trajectories being excluded), and define maps

$$(\xi, \vartheta) \in \mathscr{D} \mapsto P(\xi, \vartheta) \in (0, +\infty), E(\xi, \vartheta) \in (0, 1), Y(\xi, \vartheta) \in \mathbf{T} \tag{3.8}$$

where P, E, Y are, respectively, the parameter$/R$, the eccentricity and the argument of the pericenter for the ellipse $\mathsf{o}_{\xi\vartheta}$; here,

$$R := \text{a characteristic length of the problem} \tag{3.9}$$

(e.g., the planetary radius) . In the above, the standard parameter is divided by R to obtain a dimensionless quantity; from now on, P itself will be called the parameter and we will refer to (P, E, Y, ϑ) as the Kepler elements of $\mathsf{o}_{\xi\vartheta}$. As well known, $\mathscr{D}$ is the set of states with $\dot\vartheta > 0$ and negative unperturbed energy:

$$\mathscr{D} = \{(\xi, \vartheta) \mid \dot\vartheta > 0, \ \frac{1}{2}(\dot\rho^2 + \rho^2\dot\vartheta^2) - \frac{GM}{\rho} < 0\} \ ; \tag{3.10}$$

furthermore,

$$P(\xi, \vartheta) = \frac{\rho^4 \dot\vartheta^2}{RGM} \ , \tag{3.11}$$

$$E(\xi, \vartheta) = \sqrt{1 + \frac{2\rho^4\dot\vartheta^2}{G^2M^2}\left(\frac{1}{2}\dot\rho^2 + \frac{1}{2}\rho^2\dot\vartheta^2 - \frac{GM}{\rho}\right)} \ , \tag{3.12}$$

$$Y(\xi, \vartheta) := \text{the solution } \mathscr{Y} \in \mathbf{T} \text{ of the equation} \tag{3.13}$$

$$\rho = \frac{RP(\xi, \vartheta)}{1 + E(\xi, \vartheta)\cos(\vartheta - \mathscr{Y})} \text{ such that } \operatorname{sign}\sin(\vartheta - \mathscr{Y}) = \operatorname{sign}\dot\rho \ .$$

To continue, we introduce the notation

$$I \equiv (P, E, Y) \equiv (I^P, I^E, I^Y) \tag{3.14}$$

and write $(\xi, \vartheta) \mapsto I(\xi, \vartheta)$ for the correspondence (3.11)-(3.13). The mapping

$$\mathscr{D} \to (0, +\infty) \times (0, 1) \times \mathbf{T}^2 \ , \qquad (\xi, \vartheta) \mapsto (I(\xi, \vartheta), \vartheta) \tag{3.15}$$

is one-to-one, with inverse

$$(0, +\infty) \times (0, 1) \times \mathbf{T}^2 \to \mathscr{D} \ , \tag{3.16}$$

$$(I, \vartheta) \mapsto (\xi(I, \vartheta), \vartheta) \ ,$$

$$\dot{\rho}(I,\vartheta) := \sqrt{\frac{GM}{RP}}\, E \sin(\vartheta - Y) \ , \tag{3.17}$$

$$\dot{\vartheta}(I,\vartheta) := \sqrt{\frac{GM}{R^3 P^3}}\ (1 + E\cos(\vartheta - Y))^2 \ , \tag{3.18}$$

$$\rho(I,\vartheta) := \frac{RP}{1 + E\cos(\vartheta - Y)} \tag{3.19}$$

(see Appendix A for more details). For future use, hereafter we give the apocenter ρ_+, the pericenter ρ_- and the orbital period for the unperturbed Kepler motion on an ellipse of parameter P and eccentricity E; these are

$$\rho_\pm(P,E) = \frac{RP}{1 \mp E} \ , \qquad T_{orb}(P,E) = 2\pi\sqrt{\frac{R^3P^3}{GM(1-E^2)^3}} \ . \tag{3.20}$$

Of course, the first two equations can be inverted giving

$$P(\rho_+,\rho_-) = \frac{2\rho_+\rho_-}{R(\rho_+ + \rho_-)} \ , \qquad E(\rho_+,\rho_-) = \frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} \ . \tag{3.21}$$

3C. The equations of motions in terms of the Kepler elements. Using the transformations (3.11)–(3.19) and Eqs. (3.3) (3.4), one easily obtains a system of differential equations for (I,ϑ), namely

$$\frac{dP}{dt} = 2\varepsilon\sqrt{\frac{P}{GMR}}\, \mathcal{Q}_\vartheta(I,\vartheta) \ , \tag{3.22}$$

$$\frac{dE}{dt} = \varepsilon\sqrt{\frac{RP}{GM}}\ \sin(\vartheta - Y)\ \mathcal{Q}_\rho(I,\vartheta) \tag{3.23}$$

$$+\varepsilon\, \frac{3E + 4\cos(\vartheta - Y) + E\cos(2\vartheta - 2Y)}{2\sqrt{GMRP}}\ \mathcal{Q}_\vartheta(I,\vartheta) \ ,$$

$$\frac{dY}{dt} = -\varepsilon\sqrt{\frac{RP}{GM}}\frac{\cos(\vartheta - Y)}{E}\ \mathcal{Q}_\rho(I,\vartheta) \tag{3.24}$$

$$+\varepsilon\, \frac{\sin(\vartheta - Y)(2 + E\cos(\vartheta - Y))}{E\sqrt{GMRP}}\ \mathcal{Q}_\vartheta(I,\vartheta) \ ,$$

$$\frac{d\vartheta}{dt} = \sqrt{\frac{GM}{R^3P^3}}(1 + E\cos(\vartheta - Y))^2 \ , \tag{3.25}$$

$$\mathcal{Q}_a(I,\vartheta) := Q_a(\xi(I,\vartheta),\vartheta) \ , \quad (a \in \{\rho,\vartheta\}) \ . \tag{3.26}$$

As expected, for $Q_a = 0$ the above equations reproduce the fact that P, E, Y are constants of the motion for the unperturbed Kepler problem.

From now on, for technical reasons related to the application of the general averaging method we will think the "slow" angle Y as an element of $\mathbf{R}$ rather than $\mathbf{T}$; so, the space of Kepler elements becomes

$$\Lambda := \{ I \equiv (P, E, Y) \mid P \in (0, +\infty),\ E \in (0, 1),\ Y \in \mathbf{R} \} \ . \tag{3.27}$$

Taking this viewpoint amounts to identify the vector field defined by the above equations with its lift with respect to the projection $(P, E, Y, \vartheta) \in \Lambda \times \mathbf{T} \mapsto (P, E, [Y], \vartheta) \in (0, 1) \times (0, +\infty) \times \mathbf{T} \times \mathbf{T}$ (recall that $[\ ] : \mathbf{R} \to \mathbf{T}$ is the quotient map).

3D. From the physical time *t* to the "orbit counter" $\mathfrak{t}$. Let us consider the (maximal) solution $t \in [0, t_{\max}) \mapsto (I(t), \vartheta(t))$ of Eqs. (3.22)-(3.25), with initial data

$$I(0) = I_0 = (P_0, E_0, Y_0) \ , \qquad \vartheta(0) = 0 \ , \tag{3.28}$$

and denote by $\mathfrak{t}(\ ) : [0, t_{\max}) \mapsto \mathbf{R}$, $t \mapsto \mathfrak{t}(t)$ the unique C^1 function such that

$$\mathfrak{t}(0) = 0 \ , \qquad \vartheta(t) = [2\pi \mathfrak{t}(t)] \ . \tag{3.29}$$

Then

$$\frac{d\vartheta}{dt}(t) = 2\pi \frac{d\mathfrak{t}}{dt}(t) \ , \tag{3.30}$$

and the positivity of $d\vartheta/dt$ ensures the function $t \mapsto \mathfrak{t}(t)$ to be increasing. The image of this function is an interval $[0, \mathfrak{t}_{\max})$, and we denote the inverse function by

$$t(\ ) : [0, \mathfrak{t}_{\max}) \to [0, t_{\max}) \ , \qquad \mathfrak{t} \mapsto t(\mathfrak{t}) \ . \tag{3.31}$$

To go on, let us define

$$I(\mathfrak{t}) := [I(t)]_{t = t(\mathfrak{t})} \tag{3.32}$$

for $\mathfrak{t} \in [0, \mathfrak{t}_{\max})$. Then $dI/d\mathfrak{t} = (dI/dt)/(d\mathfrak{t}/dt) = 2\pi (dI/dt)/(d\vartheta/dt)$ and Eqs. (3.22)-(3.25) with the data (3.28) yield

$$\frac{dP}{d\mathfrak{t}} = \frac{4\pi\varepsilon P^2}{(1 + E\cos(\vartheta - Y))^2} \ \frac{R\, \mathcal{Q}_\vartheta(I, \vartheta)}{GM} \Bigg|_{\vartheta = [2\pi\mathfrak{t}]} , \qquad P(0) = P_0 \ , \tag{3.33}$$

$$\frac{dE}{d\mathfrak{t}} = \frac{2\pi\varepsilon P^2 \sin(\vartheta - Y)}{(1 + E\cos(\vartheta - Y))^2} \ \frac{R^2\, \mathcal{Q}_\rho(I, \vartheta)}{GM} \tag{3.34}$$

$$+\pi\varepsilon\, P \ \frac{3E + 4\cos(\vartheta - Y) + E\cos(2\vartheta - 2Y)}{(1 + E\cos(\vartheta - Y))^2} \ \frac{R\, \mathcal{Q}_\vartheta(I, \vartheta)}{GM} \Bigg|_{\vartheta = [2\pi\mathfrak{t}]} , \qquad E(0) = E_0 \ ,$$

$$\frac{dY}{d\mathfrak{t}} = -2\pi\varepsilon \frac{P^2 \cos(\vartheta - Y)}{E(1 + E\cos(\vartheta - Y))^2} \ \frac{R^2\, \mathcal{Q}_\rho(I, \vartheta)}{GM} \tag{3.35}$$

$$+2\pi\varepsilon\,\frac{P\sin(\vartheta-Y)(2+E\cos(\vartheta-Y)}{E(1+E\cos(\vartheta-Y))^2}\;\frac{R\,\mathcal{Q}_\vartheta(I,\vartheta)}{GM}\Bigg|_{\vartheta=[2\pi\mathfrak{t}]}\,, \qquad Y(0)=Y_0\ ;$$

we note that $\mathfrak{t}$, $R^2\,\mathcal{Q}_\rho(I,\vartheta)/(GM)$ and $R\,\mathcal{Q}_\vartheta(I,\vartheta)/(GM)$ are dimensionless quantities. Eqs. (3.33)–(3.35) are a system of the general form (2.4) for the function $\mathfrak{t}\mapsto I(\mathfrak{t})$. Eq. (3.29) indicates that $\mathfrak{t}(t)$ is the number of orbits performed by the satellite from time 0 to time t; this explains the name of *orbit counter* employed for the $\mathfrak{t}$ variable.

3E. Polar motions of a satellite around an oblate planet. From now on, the perturbation is due to a deviation of the planetary mass from the uniform distribution inside a sphere. The mass distribution is assumed to be symmetric with respect to the polar axis; O is the midpoint between the poles, and we introduce a system of Cartesian coordinates x, y, z with origin O, with the z axis (indicated by a unit vector $\boldsymbol{k}$) as polar axis ([1]). The total gravitational potential $P\mapsto V(P)$ produced by the planet admits a well known expansion in zonal harmonics, namely,

$$V(P)=-\frac{GM}{|P-O|}\left[1-\sum_{\ell=1}^{+\infty}\frac{R^\ell}{|P-O|^\ell}J_\ell P_\ell\Big(\frac{z(P)}{|P-O|}\Big)\right] \tag{3.36}$$

where P_ℓ are the Legendre polynomials, and

$$J_\ell:=-\frac{1}{MR^\ell}\int P_\ell\Big(\frac{z(Q)}{|Q-O|}\Big)\,|Q-O|^\ell\,dM(Q)\ . \tag{3.37}$$

In the above: M is total mass of the planet; R is a typical length expressing the size of the planet, usually chosen as the equatorial radius; P_ℓ are the Legendre polynomials; dM is the measure describing the mass distribution of the planet. We note that $J_1, J_2, J_3, ...$ are dimensionless due to the presence of the parameters M and R in their definitions; we will refer to them as the (dimensionless) dipole, quadrupole, octupole,... coefficients. For a derivation of (3.36) one can refer, e.g., to [6].

As for the applications of this expansion to (polar or generic) satellite motions, there is an enormous literature; we only cite the classical references [5] and [7]. To continue, let us recall that

$$P_1(\zeta):=\zeta\ , \qquad P_2(\zeta):=\frac{1}{2}(3\zeta^2-1)\ ; \tag{3.38}$$

in general $P_\ell(-\zeta)=(-1)^\ell P_\ell(\zeta)$, which implies $J_\ell=0$ if ℓ is odd and the mass distribution is reflection-invariant with respect to the equatorial plane. Often, $J_1\simeq 0$ even without an exact equatorial symmetry; in this case, the first non-negligible

[1] Of course, we are assuming m/M to be so small that O can be regarded at rest.

contribution to the expansion (3.36) is the $\ell = 2$ term. From now on, we assume $J_1 = 0$ and neglect the terms of (3.36) with $\ell \geqslant 3$; explicitating P_2, we finally obtain

$$V(P) = V_K(P) + \varepsilon W(P) \ , \tag{3.39}$$

where $V_K(P) = -GM/|P - O|$ is the Keplerian potential and

$$W(P) := -\frac{GMR^2}{|P-O|^3} \left(1 - 3\frac{z^2(P)}{|P-O|^2}\right) \ , \tag{3.40}$$

$$\varepsilon := \frac{J_2}{2} = \frac{1}{4MR^2} \int \left(|Q-O|^2 - 3z^2(Q)\right) dM(Q) \ . \tag{3.41}$$

In the sequel we always regard Eq. (3.39) as giving the exact potential in the region where the satellite is moving, and assume $\varepsilon > 0$ ([2]). The values of ε when the planet is the Earth will be reported later on.

By a *polar motion* of the satellite, we mean one with initial position and velocity in a plane Π containing the polar axis. Such a motion stays in Π for all times, since the gravitational force at any point of Π is parallel to this plane. To analyze the motion we use on Π a system of polar coordinates (ρ, ϑ), with ϑ the angle from $\boldsymbol{k}$. In these coordinates,

$$W(\rho, \vartheta) = -\frac{GMR^2}{\rho^3} \left(1 - 3 \cos^2 \vartheta\right) \ , \tag{3.42}$$

and the motion is described by Eqs. (3.3)–(3.5). A polar motion in the domain $\mathscr{D}$ of Eq. (3.10) can be described via the Kepler elements $(I, \vartheta) = (P, E, Y, \vartheta)$, and we can write down Eqs. (3.22)–(3.25) or (3.33)–(3.35) for the present choice of the perturbation. The explicit form of Eqs. (3.33)–(3.35) is

$$\frac{dP}{d\mathfrak{t}} = \varepsilon f^{P}(I, [2\pi\mathfrak{t}]), \quad \frac{dE}{d\mathfrak{t}} = \varepsilon f^{E}(I, [2\pi\mathfrak{t}]), \quad \frac{dY}{d\mathfrak{t}} = \varepsilon f^{Y}(I, [2\pi\mathfrak{t}]), \tag{3.43}$$

$$P(0) = P_0 \ , \qquad E(0) = E_0 \ , \qquad Y(0) = Y_0 \ ,$$

depending on the function

$$f = (f^i)_{i=P,E,Y} : \Lambda \times \mathbf{T} \to \mathbf{R}^3 \ , \qquad (I, \vartheta) \mapsto f(I, \vartheta) \ , \tag{3.44}$$

[2] As an example, suppose the planet is an ellipsoid $x^2/R^2 + y^2/R^2 + z^2/(\alpha R)^2 \leq 1$ and the mass distribution is uniform, i.e., $dM = (M/V)dV$ with $V = (4/3)\pi\alpha R^3$ the total volume. Then, passing to cylindrical coordinates r, φ, z we obtain

$$\varepsilon = \frac{M/V}{4MR^2} \left(\int_{-\alpha R}^{\alpha R} dz \int_0^{\sqrt{R^2 - z^2/\alpha^2}} dr \, r \int_0^{2\pi} d\varphi \, (r^2 - 2z^2) \right) = \frac{1-\alpha^2}{10} \ ,$$

which implies $\varepsilon > 0$ if $\alpha < 1$. As a supplementary remark, we mention that the analogous of W employed in [5] has a wrong sign.

$$f^{P}(I,\vartheta) := \frac{6\pi}{P}\,\Big(E\sin(\vartheta+Y) + 2\sin(2\vartheta) + E\sin(3\vartheta - Y)\Big)\ ,$$

$$f^{E}(I,\vartheta) := \frac{3\pi}{8P^2}\,\Big(E^2\sin(\vartheta-3Y) + (8+2E^2)\sin(\vartheta-Y) + (4+11E^2)\sin(\vartheta+Y)$$

$$+8E\sin(2\vartheta-2Y) + 40E\sin(2\vartheta) + 2E^2\sin(3\vartheta-3Y)$$

$$+(28+17E^2)\sin(3\vartheta-Y) + 24E\sin(4\vartheta-2Y) + 5E^2\sin(5\vartheta-3Y)\ \Big)\ ,$$

$$f^{Y}(I,\vartheta) := -\frac{3\pi}{P^2} - \frac{3\pi}{8EP^2}\,\Big(E^2\cos(\vartheta-3Y) + (8+6E^2)\cos(\vartheta-Y)$$

$$-(4-7E^2)\cos(\vartheta+Y) + 8E\cos(2\vartheta-2Y) + 24E\cos(2\vartheta) + 2E^2\cos(3\vartheta-3Y)$$

$$+(28+11E^2)\cos(3\vartheta-Y) + 24E\cos(4\vartheta-2Y) + 5E^2\cos(5\vartheta-3Y)\Big)\ .$$

This is the system to which we will apply the general scheme of Section 2.

4 Polar motions around an oblate planet: the averaging method and its error.

We take as a starting point the formulation of the problem in terms of the variables $I = (P,E,Y) \in \Lambda := (0,+\infty)\times(0,1)\times\mathbf{R}$, based on the orbit counter $\mathtt{t}$ as independent variable. So, the evolution equations have the form (2.4) with $f = (f^{P}, f^{E}, f^{Y})$ as in Eqs. (3.44). From now on, for consistency with the general notation of Sections 1 and 2, we write

$$\mathtt{I} = (\mathtt{P},\mathtt{E},\mathtt{Y}) \tag{4.1}$$

for the (maximal) solution of these equations for given initial data (we repeat that typeface symbols denote functions of $\mathtt{t}$ or $\tau = \varepsilon\mathtt{t}$; thus $\mathtt{I} : \mathtt{t} \mapsto \mathtt{I}(\mathtt{t})$ is a function, to be distinguished from a point $I = (P,E,Y)$ of the space Λ). Throughout the section, indices i,j,h,k range in $\{{\scriptstyle P},{\scriptstyle E},{\scriptstyle Y}\}$.

In the sequel, we compute the averaged equations of motion and their solutions; then we will evaluate the error of averaging with the general method of Section 2.

4A. The averaged system. The averages over $\vartheta \in \mathbf{T}$ of the functions (3.44) are $\overline{f^i}$, where, for all $I = (P,E,Y)$,

$$\overline{f}^{P}(I) = 0\ ,\quad \overline{f}^{E}(I) = 0\ ,\quad \overline{f}^{Y}(I) = -\frac{3\pi}{P^2}\ . \tag{4.2}$$

So, the averaged system for $\mathtt{J} = (\mathtt{J}^{P}, \mathtt{J}^{E}, \mathtt{J}^{Y})$ is

$$\frac{d\mathtt{J}^{P}}{d\tau} = 0,\ \frac{d\mathtt{J}^{E}}{d\tau} = 0,\ \frac{d\mathtt{J}^{Y}}{d\tau} = -\frac{3\pi}{(\mathtt{J}^{P})^2},\quad (\mathtt{J}^{P},\mathtt{J}^{E},\mathtt{J}^{Y})(0) = (P_0,E_0,Y_0) \in \Lambda\ , \tag{4.3}$$

and has solution

$$\mathsf{J}^P(\tau) = P_0 \ , \qquad \mathsf{J}^E(\tau) = E_0 \ , \qquad \mathsf{J}^Y(\tau) = Y_0 - \frac{3\pi}{{P_0}^2}\,\tau \quad \text{for } \tau \in [0, +\infty) \tag{4.4}$$

(which stays in Λ for all τ). As we see, in the approximation given by averaging, the parameter and eccentricity are constant, while the argument of the pericenter varies linearly with the rescaled time $\tau = \varepsilon \mathsf{t}$; this is a classical result.

4B. The auxiliary functions $\boldsymbol{s, v, \ldots, \frac{\partial \overline{f}}{\partial I}, \mathscr{M}, \mathscr{S}, \mathscr{G}, \mathscr{H}}$. These are required by our general method to evaluate the error of the averaging method; their definitions are found in Eqs. (2.10)–(2.14). We have computed all these functions by means of MATHEMATICA.

In the case of s, the components are

$$\begin{aligned}
s^P(I,\vartheta) &= -\frac{1}{P}\,\Big[3E\cos(\vartheta+Y) + 3\cos(2\vartheta) + E\cos(3\vartheta - Y)\Big] \ , \qquad (4.5)\\
s^E(I,\vartheta) &= -\frac{1}{16P^2}\Big[3E^2\cos(\vartheta-3Y) + (24+6E^2)\cos(\vartheta-Y) + (12+33E^2)\cos(\vartheta+Y)\\
&+ \ 12E\cos(2\vartheta-2Y) + 60E\cos(2\vartheta) + 2E^2\cos(3\vartheta-3Y) + (28+17E^2)\cos(3\vartheta-Y)\\
&+ \ 18E\cos(4\vartheta-2Y) + 3E^2\cos(5\vartheta-3Y)\Big] \ ,\\
s^Y(Y,\vartheta) &= -\frac{1}{16P^2E}\,\Big[3E^2\sin(\vartheta-3Y) + (24+18E^2)\sin(\vartheta-Y) - (12-21E^2)\sin(\vartheta+Y)\\
&+ \quad 12E\sin(2\vartheta-2Y) + 36E\sin(2\vartheta) + 2E^2\sin(3\vartheta-3Y) + (28+11E^2)\sin(3\vartheta-Y)\\
&+ \quad 18E\sin(4\vartheta-2Y) + 3E^2\sin(5\vartheta-3Y)\ \Big] \ .
\end{aligned}$$

The expressions of v, p, q, w, u are too long to report all of them here. As examples,

we write down the components v^P and u^P, which are

$$v^P(I,\vartheta)=\frac{1}{12\pi P}\Big[16E\sin Y-18E\sin(\vartheta+Y)-9\sin(2\vartheta)-2E\sin(3\vartheta-Y)\Big], \quad (4.6)$$

$$\begin{aligned}
u^P(I,\vartheta) = \ &\frac{3\pi}{128P^5}\Big[-512E\sin Y+824E^2\sin(2Y)-103E^3\sin(\vartheta-3Y) \qquad (4.7)\\
+ \ &64E^2\sin(\vartheta-2Y)-(1296E-36E^3)\sin(\vartheta-Y)-(768-1152E^2)\sin\vartheta\\
+ \ &(3524E+567E^3)\sin(\vartheta+Y)+960E^2\sin(\vartheta+2Y)-24E^3\sin(\vartheta+3Y)\\
- \ &576E^2\sin(2\vartheta-2Y)+2048E\sin(2\vartheta-Y)+(4576+1992E^2)\sin(2\vartheta)\\
+ \ &1024E\sin(2\vartheta+Y)-744E^2\sin(2\vartheta+2Y)+36E^3\sin(3\vartheta-3Y)\\
+ \ &1216E^2\sin(3\vartheta-2Y)+(3180E+521E^3)\sin(3\vartheta-Y)-(1792-640E^2)\sin(3\vartheta)\\
- \ &(1264E+172E^3)\sin(3\vartheta+Y)-27E^3\sin(3\vartheta+3Y)+560E^2\sin(4\vartheta-2Y)\\
- \ &1536E\sin(4\vartheta-Y)+(256-912E^2)\sin(4\vartheta)-336E^2\sin(4\vartheta+2Y)\\
+ \ &57E^3\sin(5\vartheta-3Y)-320E^2\sin(5\vartheta-2Y)+(288E-144E^3)\sin(5\vartheta-Y)\\
- \ &(900E+115E^3)\sin(5\vartheta+Y)+88E^2\sin(6\vartheta-2Y)-(672+696E^2)\sin(6\vartheta)\\
+ \ &4E^3\sin(7\vartheta-3Y)-(812E+133E^3)\sin(7\vartheta-Y)-328E^2\sin(8\vartheta-2Y)\\
- \ &45E^3\sin(9\vartheta-3Y)\Big] \ .
\end{aligned}$$

For future use, it is convenient to point out that

$$\begin{aligned}
\overline{p}^{\,P}(I) &= -\frac{3\pi E^2}{2P^3}\sin(2Y) \ , \qquad (4.8)\\
\overline{p}^{\,E}(I) &= \frac{3\pi}{4P^4}(10E-E^3)\sin(2Y) \ ,\\
\overline{p}^{\,Y}(I) &= \frac{3\pi}{16P^4}\Big[34+25E^2+(40+10E^2)\cos(2Y)\Big] \ ;
\end{aligned}$$

$$\frac{\partial\overline{f}^{Y}}{\partial P}(I)=\frac{6\pi}{P^3} \ , \qquad \frac{\partial\overline{f}^{\,i}}{\partial I_j}(I)=0 \qquad \text{otherwise} \ ; \qquad (4.9)$$

$$\mathscr{M}^i_{jk}(I)=0 \qquad \text{for all } i,j,k \ . \qquad (4.10)$$

To go on, we take $\mathscr{S}$ as in (2.48) and $\mathscr{G}$, $\mathscr{H}$ as in (2.20); so,

$$\mathscr{S}^i(I,\delta I,\vartheta)=\int_0^1 dx\,\frac{\partial s^i}{\partial I^j}(I+x\delta I,\vartheta); \qquad (4.11)$$

$$\mathscr{G}^i_j(I,\delta I)=\int_0^1 dx\,\frac{\partial\overline{p}^i}{\partial I^j}(I+x\delta I); \qquad (4.12)$$

$$\mathscr{H}^i_{jk}(I,\delta I)=\int_0^1 dx\,(1-x)\frac{\partial^2\overline{f}^i}{\partial I^j\partial I^k}(I+x\delta I) \ , \qquad (4.13)$$

the derivatives of s^i, $\overline{f^i}$ and $\overline{p^i}$ being computed from Eqs. (4.5) (4.2) and (4.8). In particular, we mention that

$$\frac{\partial^2 \overline{f}^{\scriptscriptstyle Y}}{\partial P^2}(I) = -\frac{18\pi}{P^4} \ , \qquad \frac{\partial^2 \overline{f}^i}{\partial I^j \partial I^k}(I) = 0 \ \text{ otherwise.} \tag{4.14}$$

4C. The functions $\mathtt{R}, \mathtt{K}$; time intervals. These functions are the solutions of Eqs. (2.21) (2.22), and can be computed in an elementary way. The expressions for $\mathtt{R}(\tau)$ and its inverse matrix are

$$\mathtt{R}(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & & \\ \dfrac{6\pi}{P_0^3}\,\tau & 0 & 1 \end{pmatrix} ; \qquad \mathtt{R}^{-1}(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & & \\ -\dfrac{6\pi}{P_0^3}\,\tau & 0 & 1 \end{pmatrix} ; \tag{4.15}$$

concerning $\mathtt{K}$, we find

$$\mathtt{K}^{\scriptscriptstyle P}(\tau) = \frac{E_0^2}{4P_0} \ \Big[\cos(2Y_0) - \cos(2Y_0 - \frac{6\pi}{P_0^2}\,\tau) \Big] \ , \tag{4.16}$$

$$\mathtt{K}^{\scriptscriptstyle E}(\tau) = -\frac{10E_0 - E_0^3}{8P_0^2} \Big[\cos(2Y_0) - \cos(2Y_0 - \frac{6\pi}{P_0^2}\,\tau) \Big] \ ,$$

$$\mathtt{K}^{\scriptscriptstyle Y}(\tau) = \frac{3\pi}{16P_0^4} \Big[34 + 25E_0^2 + 8E_0^2\cos(2Y_0) \Big] \tau + \frac{20 + E_0^2}{16P_0^2} \Big[\sin(2Y_0) - \sin(2Y_0 - \frac{6\pi}{P_0^2}\,\tau) \Big] \ .$$

In the above, $\tau \in [0, +\infty)$; however, from now on we put a limitation $\tau \in [0, U)$, for some finite U; consequently, the "orbit counter" $\mathtt{t}$ will range in $[0, U/\varepsilon)$.

4D. The auxiliary functions $\boldsymbol{\rho^i, a^i_{(0)}, a^i_j}$, ..., $\boldsymbol{e^i_{jk}}$, $\boldsymbol{\alpha^i}$, $\boldsymbol{\gamma^i}$. These functions must be constructed so as to fulfill the inequalities in Section 2. First of all, we must find a function $\rho = (\rho^i) \in C([0, U), [0, +\infty]^3)$ such that $B(\mathtt{J}(\tau), \rho(\tau)) \subset \Lambda$ for all τ, with Λ as in Eq. (3.27). The cited equation can be satisfied putting

$$\rho^{\scriptscriptstyle P}(\tau) := P_0 \ , \quad \rho^{\scriptscriptstyle E}(\tau) := \min(E_0, 1 - E_0) \ , \qquad \rho^{\scriptscriptstyle Y}(\tau) := +\infty \tag{4.17}$$

for $\tau \in [0, U)$ (recall that $\mathtt{J}^{\scriptscriptstyle P}(\tau) = P_0$ and $\mathtt{J}^{\scriptscriptstyle E}(\tau) = E_0$). From these functions, we define Γ_ρ as in Eq. (2.25); this has elements $(\tau, r) = (\tau, r^{\scriptscriptstyle P}, r^{\scriptscriptstyle E}, r^{\scriptscriptstyle Y})$.
To go on, we must find a system of auxiliary functions

$$a^i_{(0)} \in C^\infty([0, U), [0, +\infty)) \ , \quad a^i_j, b^i, c^i, d^i_j, e^i_{jk} \in C^\infty(\Gamma_\rho, [0, +\infty)) \tag{4.18}$$

such that Eqs. (2.50) (2.51) and (2.27)–(2.30) be satisfied. As explained in subsection 2F, each function $a^i_{(0)}$ should give an accurate upper bound for the left-hand side

of Eq. (2.50); for this reason we have developed an algorithm based on numerical maximization over ϑ and subsequent interpolation in τ, see Appendix B.

Let us pass to the functions $a^i_j, b^i, ..., e^i_{jk}$; we refer again to subsection 2F, suggesting to find these functions by fairly rough majorizations of the left-hand sides of Eqs. (2.51) and (2.27)–(2.30). The expressions to be bounded are very lengthy in some cases; they were majorized with the method described in Appendix C (sometimes using the symbolic mode of MATHEMATICA for the necessary computations). Here, we only report the final expressions of the majorizing functions, which are in fact τ-independent; for this reason we write $a^i_j(r)$, $b^i(r)$, etc., instead of $a^i_j(\tau, r)$, $b^i(\tau, r)$, etc.. It should also be noted that the dependence on r is through the components r^P, r^E. In the sequel, for the sake of brevity we put

$$P_\pm := P_0 \pm r^P \ , \qquad E_\pm := E_0 \pm r^E \ ; \tag{4.19}$$

with these notations, we can take

$$\begin{aligned}
a^P_P(r) &:= \frac{3+4E_+}{P_-^2} \ , & a^P_E(r) &:= \frac{4}{P_-} \ , & a^P_Y(r) &:= \frac{4E_+}{P_-} \ , \\
a^E_P(r) &:= \frac{32+45E_+ + 32E_+^2}{4P_-^3} \ , & a^E_E(r) &:= \frac{45+64E_+}{8P_-^2} \ , & a^E_Y(r) &:= \frac{16+15E_+ + 20E_+^2}{4P_-^2} \ , \\
a^Y_P(r) &:= \frac{32+33E_+ + 29E_+^2}{4P_-^3 E_-} \ , & a^Y_E(r) &:= \frac{32+29E_+^2}{8P_-^2 E_-^2} \ , & a^Y_Y(r) &:= \frac{32+30E_+ + 37E_+^2}{8P_-^2 E_-} \ ;
\end{aligned} \tag{4.20}$$

$$b^P(r) := \frac{54+112E_+ + 33E_+^2}{8P_-^3} \ , \tag{4.21}$$

$$b^E(r) := \frac{6112+10832E_+ + 6940E_+^2 + 11372E_+^3 + 1441E_+^4}{512P_-^4 E_-} \ ,$$

$$\begin{aligned}
b^Y(r) := {} & \frac{1}{256P_0^3 E_-^2 P_-^4} \Big[3520P_0^3 + 16384P_0^3 E_+ + 9340P_0^3 E_+^2 \\
& + 8940P_0^3 E_+^3 + 1861P_0^3 E_+^4 + 1152E_+^2 P_+^3 + 4608E_+^3 P_+^3\Big] \ ;
\end{aligned}$$

$$c^P(r) := 3\pi \frac{504+1024E_+ + 713E_+^2 + 124E_+^3}{8P_-^5} \ , \tag{4.22}$$

$$\begin{aligned}
c^E(r) := {} & \frac{3\pi}{2048P_-^6 E_-^2} \Big[148736 + 738384E_+ + 1062656E_+^2 \\
& + 1220344E_+^3 + 675146E_+^4 + 336591E_+^5 + 26855E_+^6\Big] \ ,
\end{aligned}$$

$$\begin{aligned}
c^Y(r) := {} & \frac{\pi}{1024P_0^3 E_-^3 P_-^6} \Big[370944P_0^3 + 2214336P_0^3 E_+ + 5434752P_0^3 E_+^2 \\
& + 4927104P_0^3 E_+^3 + 2945040P_0^3 E_+^4 + 1225668P_0^3 E_+^5 + 147777P_0^3 E_+^6 \\
& + 231936E_+^3 P_+^3 + 442368E_+^4 P_+^3 + 196608E_+^5 P_+^3\Big] \ ;
\end{aligned}$$

$$d^{P}_{P}(r) := \frac{9\pi E_+^2}{2P_-^4}\ , \qquad d^{P}_{E}(r) := \frac{3\pi E_+}{P_-^3}\ , \qquad d^{P}_{Y}(r) := \frac{3\pi E_+^2}{P_-^3}\ , \tag{4.23}$$

$$d^{E}_{P}(r) := \frac{3\pi E_+(10+E_+^2)}{P_-^5}\ ,\ d^{E}_{E}(r) := \frac{3\pi(10+3E_+^2)}{4P_-^4}\ ,\ d^{E}_{Y}(r) := \frac{3\pi E_+(10+E_+^2)}{2P_-^4}\ ,$$

$$d^{Y}_{P}(r) := \frac{3\pi(74+35E_+^2)}{4P_-^5}\ ,\ d^{Y}_{E}(r) := \frac{105\pi E_+}{8P_-^4}\ , \quad d^{Y}_{Y}(r) := \frac{15\pi(4+E_+^2)}{4P_-^4}\ ;$$

$$e^{Y}_{PP}(r) := \frac{18\pi}{P_-^4}\ , \qquad e^{i}_{jk}(\tau,r) := 0 \quad \text{otherwise}\ . \tag{4.24}$$

From the above functions we obtain a set of C^∞ functions α^i, γ^i, specializing the prescriptions (2.33) (2.34) to the present case. Explicitly,

$$\alpha^i(\tau,r) := a^i(\tau,r) + \varepsilon b^i(r) = a^i_{(0)}(\tau) + a^i_k(r) r^k + \varepsilon b^i(r)\ , \tag{4.25}$$

$$\gamma^i(\tau,r,\ell) \equiv \gamma^i(r,\ell) := c^i(r) + d^i_j(r)\ell^j + \frac{1}{2} e^i_{jk}(r)\ell^j\ell^k\ , \tag{4.26}$$

for $(\tau,r) \in \Gamma_\rho$ and $\ell \in [0,+\infty)^3$.

4E. The matrix $(1-\varepsilon\partial\alpha/\partial r)$ and its inverse. From Eq. (4.25), one finds that the derivatives of α^i with respect to the r variables depend on r but not on τ. From the same equation, we see that our matrix has components

$$\delta^i_j - \varepsilon\frac{\partial\alpha^i}{\partial r^j}(r) = \delta^i_j - \varepsilon\Big(\frac{\partial a^i_k}{\partial r^j}(r) r^k + a^i_j(r) + \varepsilon\frac{\partial b^i}{\partial r^j}(r)\Big), \qquad (i,j = P,E,Y)\ ; \tag{4.27}$$

all the derivatives in the right hand side are easily obtained from Eq.s (4.20) (4.21). Eq. (2.42), that is basic in our approach, contains the inverse of this matrix; this could be written explicitly, but its expression is uselessly complicated. For this reason, in the subsequent numerical computations we use an approximation giving the inverse up to a third order error $O_3(\varepsilon,r)$, for $(\varepsilon,r) = (\varepsilon, r^p, r^E, r^Y) \to 0$ ([3]). Neglecting this error, we have

$$\Big(1-\varepsilon\frac{\partial\alpha}{\partial r}(r)\Big)^{-1} = 1 + \varepsilon M + \varepsilon r^k N_{(k)} + \varepsilon^2 Q\ , \tag{4.28}$$

(with a sum for $k \in \{P,E,Y\}$), where we have introduced the matrices

$$M := \begin{pmatrix} \dfrac{3+4E_0}{P_0^2} & \dfrac{4}{P_0} & \dfrac{4E_0}{P_0} \\ \dfrac{32+45E_0+32E_0^2}{4P_0^3} & \dfrac{45+64E_0}{8P_0^2} & \dfrac{16+15E_0+20E_0^2}{4P_0^2} \\ \dfrac{32+33E_0+29E_0^2}{4E_0P_0^3} & \dfrac{32+29E_0^2}{8E_0^2P_0^2} & \dfrac{32+30E_0+37E_0^2}{8E_0P_0^2} \end{pmatrix}\ ; \tag{4.29}$$

[3] Recall that, in Eq. (2.42), the matrix in which we are interested is evaluated with $r^i = \varepsilon\mathfrak{n}^i(\tau)$; with this position for r, (4.28) gives an inverse up to $O(\varepsilon^3)$.

$$N_{(P)} := \begin{pmatrix} \dfrac{4(3+4E_0)}{P_0^3} & \dfrac{8}{P_0^2} & \dfrac{4E_0}{P_0^2} \\ \dfrac{3(32+45E_0+32E_0^2)}{2P_0^4} & \dfrac{45+64E_0}{2P_0^3} & \dfrac{16+15E_0+20E_0^2}{2P_0^3} \\ \dfrac{3(32+33E_0+29E_0^2)}{2E_0P_0^4} & \dfrac{32+33E_0+58E_0^2}{2E_0^2P_0^3} & \dfrac{32+30E_0+37E_0^2}{4E_0P_0^3} \end{pmatrix} ,$$

$$N_{(E)} := \begin{pmatrix} \dfrac{8}{P_0^2} & 0 & \dfrac{4}{P_0} \\ \dfrac{45+64E_0}{2P_0^3} & \dfrac{16}{P_0^2} & \dfrac{5(3+8E_0)}{4P_0^2} \\ \dfrac{32+33E_0+58E_0^2}{2E_0^2P_0^3} & \dfrac{16+29E_0^2}{E_0^3P_0^2} & \dfrac{32+60E_0+111E_0^2}{8E_0^2P_0^2} \end{pmatrix} ,$$

$$N_{(Y)} := \begin{pmatrix} \dfrac{4E_0}{P_0^2} & \dfrac{4}{P_0} & 0 \\ \dfrac{16+15E_0+20E_0^2}{2P_0^3} & \dfrac{5(3+8E_0)}{4P_0^2} & 0 \\ \dfrac{32+30E_0+37E_0^2}{4E_0P_0^3} & \dfrac{32+60E_0+111E_0^2}{8E_0^2P_0^2} & 0 \end{pmatrix} ; \qquad (4.30)$$

$$Q := \begin{pmatrix} \dfrac{746+1152E_0+715E_0^2}{8P_0^4} & \dfrac{64+194E_0+283E_0^2}{4E_0P_0^3} & \dfrac{64+84E_0+109E_0^2}{2P_0^3} \\ \dfrac{\mathtt{Q}_P^E}{128E_0P_0^5} & \dfrac{\mathtt{Q}_E^E}{512E_0^2P_0^4} & \dfrac{\mathtt{Q}_Y^E}{32E_0P_0^4} \\ \dfrac{\mathtt{Q}_P^Y}{64E_0^2P_0^5} & \dfrac{\mathtt{Q}_E^Y}{128E_0^3P_0^4} & \dfrac{\mathtt{Q}_Y^Y}{64E_0^2P_0^4} \end{pmatrix} ,$$

$$\begin{aligned} \mathtt{Q}_P^E &:= 10208+27728E_0+44440E_0^2+46244E_0^3+18369E_0^4 , \\ \mathtt{Q}_E^E &:= 14304+29344E_0+71068E_0^2+121568E_0^3+65637E_0^4 , \\ \mathtt{Q}_Y^E &:= 512+1680E_0+4405E_0^2+4455E_0^3+3044E_0^4 , \\ \mathtt{Q}_P^Y &:= 7616+24832E_0+25096E_0^2+27300E_0^3+7719E_0^4 , \\ \mathtt{Q}_E^Y &:= 5568+29376E_0+33400E_0^2+42444E_0^3+15153E_0^4 , \\ \mathtt{Q}_Y^Y &:= 2048+2880E_0+7524E_0^2+5202E_0^3+4385E_0^4 . \end{aligned} \qquad (4.31)$$

For more details on the approximate inverse (4.28), see Appendix D.

4F. The functions R^i_j, P^i_j. According to Eq. (2.35), these should fulfill the inequalities

$$|\mathtt{R}(\tau)|^i_j \leqslant R^i_j(\tau) , \quad |\mathtt{R}^{-1}(\tau)|^i_j \leqslant P^i_j(\tau) \quad \text{for } \tau \in [0, U) . \qquad (4.32)$$

On account of Eq. (4.15), we can take

$$\big(R^i_j(\tau)\big) := \big(P^i_j(\tau)\big) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \\ \dfrac{6\pi}{P_0^3}\,\tau & 0 & 1 \end{pmatrix} . \tag{4.33}$$

4G. The main result: the estimates $|\boldsymbol{L^i(t)}| \leqslant \boldsymbol{\mathfrak{n}^i(\varepsilon t)}$ from Proposition 2.1. The "$\mathscr{N}$-operation". We are finally ready to discuss the difference

$$\mathtt{L}(t) := \frac{1}{\varepsilon}\,[\mathtt{I}(t) - \mathtt{J}(\varepsilon t)] = \frac{1}{\varepsilon}\Big(\mathtt{P}(t) - P_0, \mathtt{E}(t) - E_0, \mathtt{Y}(t) - \mathtt{J}^{\scriptscriptstyle Y}(\varepsilon t)\Big) \tag{4.34}$$

where $\mathtt{I} = (\mathtt{P}, \mathtt{E}, \mathtt{Y})$ is the solution of Eqs. (3.43). We follow the scheme of Proposition 2.1; this requires to determine $\ell_0 = (\ell_0^{\scriptscriptstyle P}, \ell_0^{\scriptscriptstyle E}, \ell_0^{\scriptscriptstyle Y})$ by solving a fixed point problem

$$\alpha^i(0, \varepsilon\ell_0) = \ell_0^i \ . \tag{4.35}$$

In the examples that follow ℓ_0 is always found numerically, as explained in Remark 2.2(a). After ℓ_0 has been found, we must solve (again numerically) a Cauchy problem for the unknown functions $\mathfrak{m} = (\mathfrak{m}^i), \mathfrak{n} = (\mathfrak{n}^i) \in C^1([0,U), \mathbf{R}^3)$, i.e.,

$$\frac{d\mathfrak{m}^i}{d\tau} = P^i_j\,\gamma^j(\varepsilon\mathfrak{n}, \mathfrak{n}) \ , \qquad \mathfrak{m}^i(0) = 0 \ , \tag{4.36}$$

$$\frac{d\mathfrak{n}^i}{d\tau} = \Big(1 - \varepsilon\frac{\partial\alpha}{\partial r}\,(\cdot, \varepsilon\mathfrak{n})\Big)^{-1,i}_{\quad\ k}\left(\frac{\partial\alpha^k}{\partial\tau}\,(\cdot, \varepsilon\mathfrak{n}) + \varepsilon R^k_h P^h_j\,\gamma^j(\varepsilon\mathfrak{n}, \mathfrak{n}) + \varepsilon\frac{dR^k_j}{d\tau}\mathfrak{m}^j\right) \ , \tag{4.37}$$

$$\mathfrak{n}^i(0) = \ell_0^i$$

with the domain conditions (2.43), having in this case the form

$$0 < \varepsilon\mathfrak{n}^{\scriptscriptstyle P} < P_0 \ , \quad 0 < \varepsilon\mathfrak{n}^{\scriptscriptstyle E} < \min(E_0, 1 - E_0) \ , \quad \mathfrak{n}^{\scriptscriptstyle Y} > 0 \ ,$$

$$\det\Big(1 - \varepsilon\frac{\partial\alpha}{\partial r}\,(\cdot, \varepsilon\mathfrak{n})\Big) > 0 \ . \tag{4.38}$$

If the above problem has solution on $[0,U)$, our general framework grants that the solution $\mathtt{I} = (\mathtt{P}, \mathtt{E}, \mathtt{Y})$ of (3.43) exists on $[0, U/\varepsilon)$, and gives the bounds

$$|\mathtt{L}^i(t)| \leqslant \mathfrak{n}^i(\varepsilon t) \qquad \text{for } t \in [0, U/\varepsilon) \ , \tag{4.39}$$

i.e.,

$$|\,\mathtt{P}(t) - P_0| \leqslant \varepsilon\mathfrak{n}^{\scriptscriptstyle P}(\varepsilon t), \quad |\,\mathtt{E}(t) - E_0| \leqslant \varepsilon\mathfrak{n}^{\scriptscriptstyle E}(\varepsilon t), \quad |\,\mathtt{Y}(t) - \mathtt{J}^{\scriptscriptstyle Y}(\varepsilon t)| \leqslant \varepsilon\mathfrak{n}^{\scriptscriptstyle Y}(\varepsilon t). \tag{4.40}$$

With a slight variation of the terminology of [1] [2], we call "$\mathscr{N}$-operation" the execution of *all* the numerical computations required by the present approach to obtain the estimators $(\mathfrak{n}^i)$, given the initial data P_0, E_0, E_0 and the values of ε, U. These computations include:

(a) the numerical evaluations and subsequent interpolations, necessary to determine the functions $a^i_{(0)}$ $(i = {\scriptscriptstyle P}, {\scriptscriptstyle E}, {\scriptscriptstyle Y})$ on $[0, U)$. We have already mentioned Appendix B, that describes everything in detail;

(b) the determination of the fixed point ℓ_0;

(c) the numerical solution of the Cauchy problem (4.36) (4.37) for $(\mathfrak{m}^i)$, $(\mathfrak{n}^i)$.

From now on, we will indicate with $\mathfrak{T}_{\mathscr{N}}$ the CPU time required to perform (a)(b)(c); this time depends on the chosen hardware and software, but is anyhow useful for comparison with other computational approaches, such as the one described hereafter.

4H. The "$\mathfrak{L}$-operation": a way to test the efficiency of the previous estimates. This operation was already considered in [2], with a role similar to the one that we ascribe to it here: checking the reliability of the $\mathscr{N}$- estimates by a direct numerical treatment of the system (3.43), for $t \in [0, U/\varepsilon)$. The direct attack of (3.43) is possible when the time scale U/ε is not overwhelmingly large; for larger U/ε, the $\mathfrak{L}$-operation is too expensive, and this is just the case where the $\mathscr{N}$-procedure becomes more useful.

The direct attack to (3.43) is performed substituting in these equations $\mathtt{P}(t) = P_0 + \varepsilon \mathtt{L}^{\scriptscriptstyle P}(t)$, $\mathtt{E}(t) = E_0 + \varepsilon \mathtt{L}^{\scriptscriptstyle E}(t)$, $\mathtt{Y}(t) = \mathtt{J}^{\scriptscriptstyle Y}(\varepsilon t) + \varepsilon \mathtt{L}^{\scriptscriptstyle Y}(t)$, which gives rise to the equations

$$\frac{d\mathtt{L}^{\scriptscriptstyle P}}{dt}(t) = f^{\scriptscriptstyle P}(P_0 + \varepsilon \mathtt{L}^{\scriptscriptstyle P}(t), E_0 + \varepsilon \mathtt{L}^{\scriptscriptstyle E}(t), \mathtt{J}^{\scriptscriptstyle Y}(\varepsilon t) + \varepsilon \mathtt{L}^{\scriptscriptstyle Y}(t)), \tag{4.41}$$

$$\frac{d\mathtt{L}^{\scriptscriptstyle E}}{dt}(t) = f^{\scriptscriptstyle E}(P_0 + \varepsilon \mathtt{L}^{\scriptscriptstyle P}(t), E_0 + \varepsilon \mathtt{L}^{\scriptscriptstyle E}(t), \mathtt{J}^{\scriptscriptstyle Y}(\varepsilon t) + \varepsilon \mathtt{L}^{\scriptscriptstyle Y}(t)),$$

$$\frac{d\mathtt{L}^{\scriptscriptstyle Y}}{dt}(t) = f^{\scriptscriptstyle Y}(P_0 + \varepsilon \mathtt{L}^{\scriptscriptstyle P}(t), E_0 + \varepsilon \mathtt{L}^{\scriptscriptstyle E}(t), \mathtt{J}^{\scriptscriptstyle Y}(\varepsilon t) + \varepsilon \mathtt{L}^{\scriptscriptstyle Y}(t)) + \frac{3\pi}{\mathtt{J}^{\scriptscriptstyle P}(\varepsilon t)^2} ,$$

$$(\mathtt{L}^{\scriptscriptstyle P}, \mathtt{L}^{\scriptscriptstyle E}, \mathtt{L}^{\scriptscriptstyle Y})(0) = 0 .$$

By definition, the $\mathfrak{L}$-operation is the numerical solution of this Cauchy problem in the unknowns $\mathtt{L}^i$, for $t \in [0, U/\varepsilon)$; $\mathfrak{T}_{\mathfrak{L}}$ will indicate the necessary CPU time.

5 Examples.

Introducing the examples. In the case of the Earth,

$$GM = 3.98600442 \cdot 10^{14} \frac{\mathrm{m}^3}{\mathrm{sec}^2} , \qquad R = 6.378135 \cdot 10^6 \, \mathrm{m} , \tag{5.1}$$

$$\varepsilon = 5.457 \cdot 10^{-4} \ .$$

So, for a satellite on a polar orbit around the Earth, the unperturbed apogee and perigee ρ_+, ρ_- and the orbital period T_{orb} are determined in this way by the initial data (P_0, E_0, Y_0):

$$\rho_\mp = \frac{P_0}{1 \pm E} \times 6.378135 \times 10^3 \, \mathrm{Km}, \quad T_{orb} = \frac{P_0^{3/2}}{(1 - E_0^2)^{3/2}} \times 1.408150 \, \mathrm{hours}.$$

In the sequel we present two examples, with initial data (P_0, E_0, Y_0) very similar to the actual data of two real satellites (Polar and Cos-B). The MATHEMATICA package, already mentioned in relation to the symbolic treatment of the problem, has been employed (on a PC) for the required numerical computations.

We repeat a comment of the Introduction: the examples are not fully realistic since they do not account for perturbations different from the J_2 gravitational term; nevertheless, we think they have some interest because they show the effectiveness of the method, suggesting that our approach could be applied as well including other perturbations.

Each example is worked out along these lines:

(i) Firstly, we take $U/\varepsilon = 3000$. We perform both the $\mathscr{N}$- and the $\mathfrak{L}$-operations, using the second to test the first one. We give figures reporting the graphs of $\varepsilon|\,\mathtt{L}^i(t)|$ (rapidly oscillating) and $\varepsilon\mathfrak{n}^i(\varepsilon t)$, for $i \in \{{\scriptscriptstyle P}, {\scriptscriptstyle E}, {\scriptscriptstyle Y}\}$ and $t \in [0, U/\varepsilon)$; the CPU times $\mathfrak{T}_{\mathscr{N}}$, $\mathfrak{T}_{\mathfrak{L}}$ are also indicated.

(ii) Next, we choose $U/\varepsilon = 60000$. The $\mathscr{N}$-operation is still performed within short CPU times; we give figures reporting $\mathfrak{T}_{\mathscr{N}}$ and the graphs of the estimators $(\varepsilon\mathfrak{n}^i)$. On the contrary, the $\mathfrak{L}$-operation exceeds the capabilities of the machine employed.

Example 1. We take

$$P_0 := 3.000 \ , \qquad E_0 := 0.6640 \ , \qquad Y_0 := 0.0000 \ , \tag{5.2}$$

corresponding to

$$\rho_+ = 56950 \, \mathrm{Km} \ , \qquad \rho_- = 11500 \, \mathrm{Km} \ , \qquad T_{orb} = 17.50 \, \mathrm{hours} \ . \tag{5.3}$$

These data are very similar to the initial conditions of the Polar satellite, taking $t = 0$ on April 1997 [8]. Eq. (4.4) for the solution of the averaged system on any interval $[0, U)$ gives $\mathtt{J}^{\scriptscriptstyle P}(\tau) = \mathrm{const.} = P_0$, $\mathtt{J}^{\scriptscriptstyle E}(\tau) = \mathrm{const.} = E_0$, and

$$\mathtt{J}^{\scriptscriptstyle Y}(\tau) = -1.047\tau \ . \tag{5.4}$$

Figures 1a, 1b and 1c report the graphs of $\mathfrak{n}^i(\varepsilon t)$ and $|\mathtt{L}^i(t)|$ for $t \in [0, 3000]$, as produced by the $\mathfrak{L}$- and $\mathscr{N}$-operations. We note that $3000\, T_{orb} \simeq 6$ years. The CPU times (in seconds) for $\mathscr{N}$ and $\mathfrak{L}$ reported in Fig.1a refer to the overall calculation, also including the ${\scriptscriptstyle E}$ and ${\scriptscriptstyle Y}$ components plotted in Figures 1b, 1c.

Figures 1d, 1e and 1f report the graphs of $\mathfrak{n}^i(\varepsilon t)$ and $|\,\mathtt{L}^i(t)|$ for $t \in [0, 60000]$, as produced by the $\mathscr{N}$-operations. We note that $60000\, T_{orb} \simeq 120$ years. The CPU time indicated in Fig.1d also includes calculations of the $_E$ and $_Y$ components, plotted in Figures 1e, 1f. In comparison with the previous case $t \in [0, 3000]$, the number of orbits is increased by a factor 20; in spite of this, $T_{\mathscr{N}}$ changes by a factor $\lesssim 2$.

Example 2. We take

$$P_0 := 1.973 \;, \qquad E_0 = 0.8817 \;, \qquad Y_0 := 0.9600 \;, \tag{5.5}$$

corresponding to

$$\rho_+ = 106400\,\mathrm{Km} \;, \qquad \rho_- = 6688\,\mathrm{Km} \;, \qquad T_{orb} = 37.16\,\mathrm{hours} \;. \tag{5.6}$$

These data are very similar to the initial conditions at launch (August 1975) of the Cos-B satellite [9]. Eq. (4.4) for the solution of the averaged system on any interval $[0, U)$ gives $\mathtt{J}^P(\tau) = \mathrm{const.} = P_0$, $\mathtt{J}^E(\tau) = \mathrm{const.} = E_0$ and

$$\mathtt{J}^Y(\tau) = 0.9600 - 2.421\,\tau \;. \tag{5.7}$$

All figures for this example give the same information as the corresponding ones of Example 1. In particular, Figures 2a, 2b and 2c refer to computations for $t \in [0, 3000]$, while Figures 2d, 2e, 2f refer to the case $t \in [0, 60000]$. We note that $3000\, T_{orb} \simeq 12$ years and $60000\, T_{orb} \simeq 250$ years. The CPU times (in seconds) for $\mathscr{N}$ and $\mathfrak{L}$ are similar to the ones in Example 1, so the comments given therein also apply to this case.

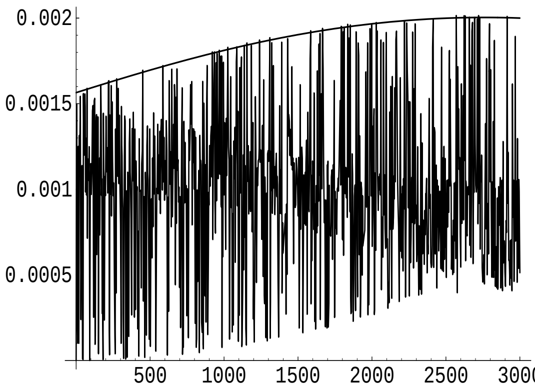

Figure 1a. Example 1, for $t \in [0, 3000]$. Total CPU times: $T_{\mathscr{N}} = 5.18$ sec, $T_{\mathfrak{L}} = 29.08$ sec. Graphs of $\varepsilon \mathfrak{n}^P(\varepsilon t)$ and $\varepsilon|\mathtt{L}^P(t)|$.

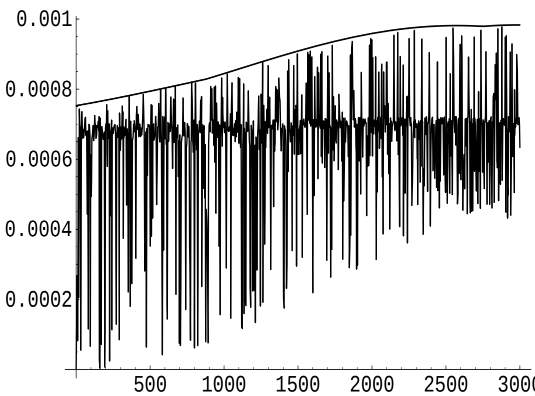

Figure 1b. The same situation as in Fig. 1a. Graphs of $\varepsilon \mathfrak{n}^E(\varepsilon t)$ and $\varepsilon|\mathtt{L}^E(t)|$.

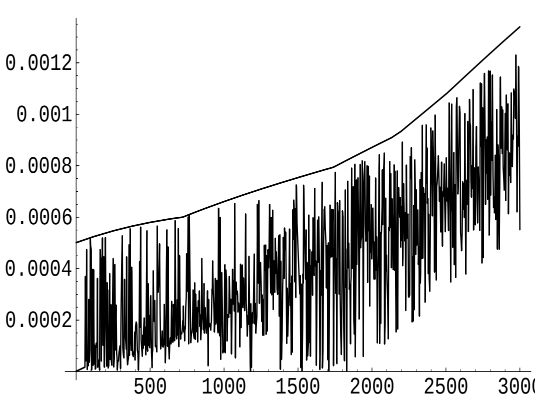

Figure 1c. The same situation as in Fig. 1a. Graphs of $\varepsilon \mathfrak{n}^Y(\varepsilon t)$ and $\varepsilon|\mathtt{L}^Y(t)|$.

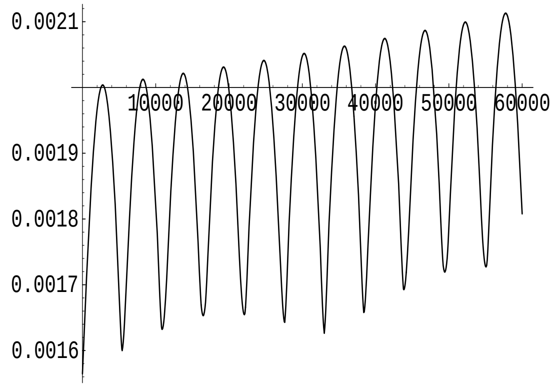

Figure 1d. Example 1, for $t \in [0, 60000]$. Total CPU time: $T_{\mathscr{N}} = 10.11$ sec. Graph of $\varepsilon \mathfrak{n}^P(\varepsilon t)$.

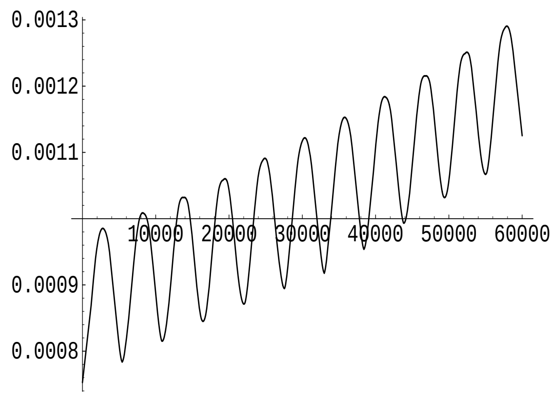

Figure 1e. The same situation as in Fig. 1d. Graph of $\varepsilon \mathfrak{n}^E(\varepsilon t)$.

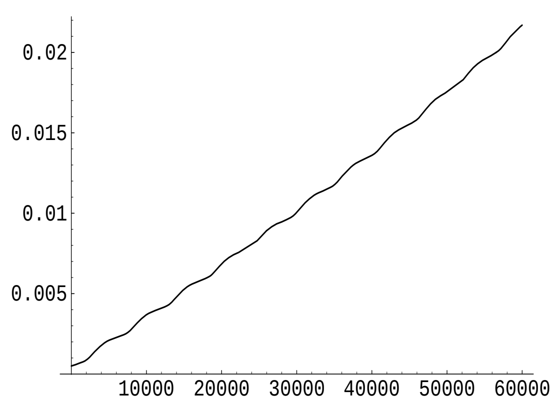

Figure 1f. The same situation as in Fig. 1d. Graph of $\varepsilon \mathfrak{n}^Y(\varepsilon t)$. (Note that $\mathtt{J}^Y(\varepsilon t) = -34.29$ for $t = 60000$).

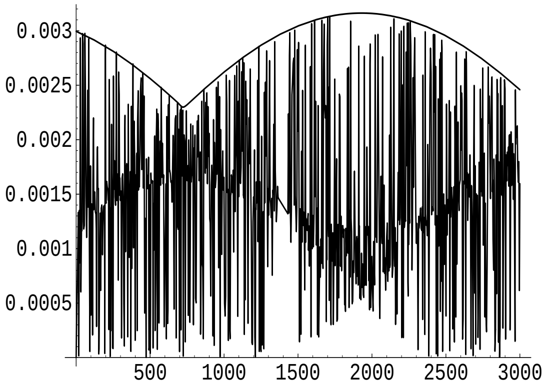

Figure 2a. Example 2, for $t \in [0, 3000]$. Total CPU times: $T_{\mathscr{N}} = 5.77$ sec, $T_{\mathfrak{L}} = 33.02$ sec. Graphs of $\varepsilon\mathfrak{n}^P(\varepsilon t)$ and $\varepsilon|\mathtt{L}^P(t)|$.

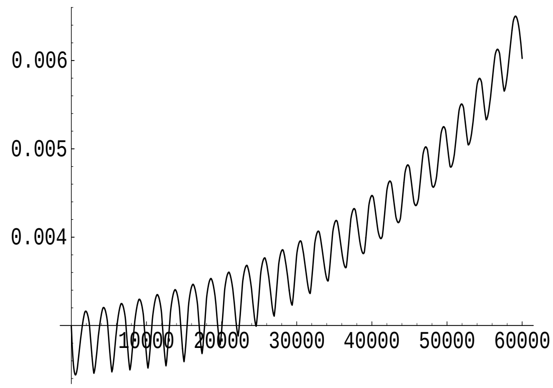

Figure 2d. Example 2, for $t \in [0, 60000]$. Total CPU time: $T_{\mathscr{N}} = 11.28$ sec. Graph of $\varepsilon\mathfrak{n}^P(\varepsilon t)$.

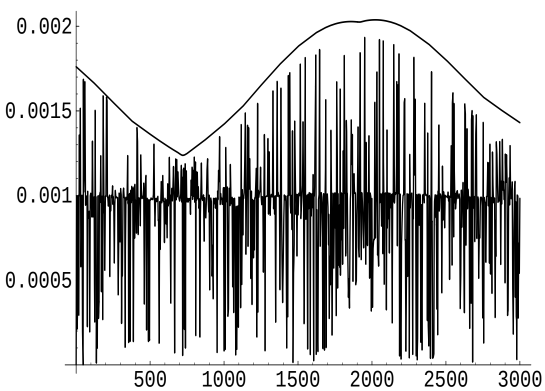

Figure 2b. The same situation as in Fig. 2a. Graphs of $\varepsilon\mathfrak{n}^E(\varepsilon t)$ and $\varepsilon|\mathtt{L}^E(t)|$.

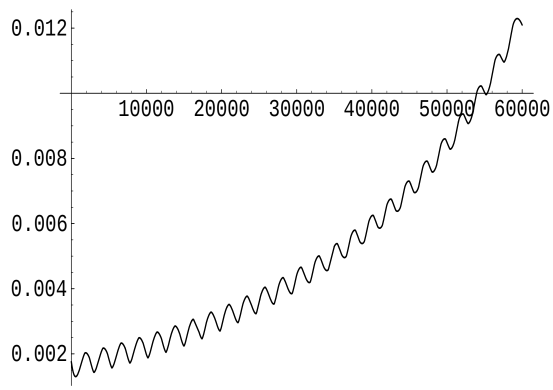

Figure 2e. The same situation as in Fig. 2d. Graph of $\varepsilon\mathfrak{n}^E(\varepsilon t)$.

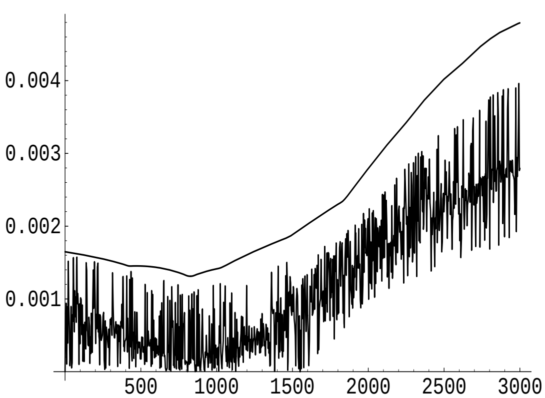

Figure 2c. The same situation as in Fig. 2a. Graphs of $\varepsilon\mathfrak{n}^Y(\varepsilon t)$ and $\varepsilon|\mathtt{L}^Y(\varepsilon t)|$.

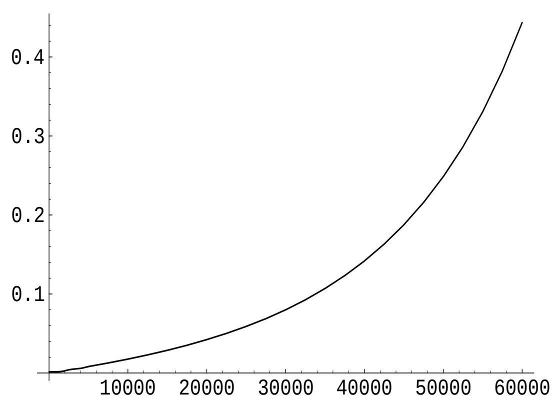

Figure 2f. The same situation as in Fig. 2d. Graph of $\varepsilon\mathfrak{n}^Y(\varepsilon t)$. (Note that $\mathtt{J}^Y(\varepsilon t) = -78.31$ for $t = 60000$)

A Appendix. On Kepler elements.

We refer to the framework of subsection 3B, having fixed $(\xi, \vartheta) = (\dot{\rho}, \dot{\vartheta}, \rho, \vartheta) \in \mathscr{D}$. The unperturbed Kepler orbit $\mathsf{o}_{\xi\vartheta}$ with these initial data at time $t = 0$ is an ellipse with (dimensionless) parameter $P \equiv P(\xi, \vartheta)$, eccentricity $E \equiv E(\xi, \vartheta)$ and argument of the pericenter $Y \equiv Y(\xi, \vartheta)$. In any textbook on classical mechanics, one finds the equations

$$\rho = \frac{RP}{1 + E\cos(\vartheta - Y)} \ , \tag{A.1}$$

$$\dot{\rho} = \frac{RPE\sin(\vartheta - Y)}{(1 + E\cos(\vartheta - Y))^2}\dot{\vartheta} \ , \tag{A.2}$$

$$RP = \frac{\rho^4\dot{\vartheta}^2}{GM} \ , \tag{A.3}$$

$$E = \sqrt{1 + \frac{2\rho^4\dot{\vartheta}^2}{G^2M^2}\left(\frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\rho^2\dot{\vartheta}^2 - \frac{GM}{\rho}\right)} \ . \tag{A.4}$$

Eq. (A.1) is the representation of a conic in polar coordinates; Eq. (A.2) follows taking the derivative of this polar representation with respect to time along the unperturbed Kepler motion, i.e., regarding P, E, Y as time independent. Eqs. (A.3) (A.4) give the parameter and the eccentricity as functions of the angular momentum and the total energy (per unit mass) $\rho^2\dot{\vartheta}$ and $(1/2)\dot{\rho}^2 + (1/2)\rho^2\dot{\vartheta}^2 - GM/\rho$.

We note that Eq. (A.1) and Eqs. (A.3) (A.4) are equivalent to Eq. (A.1), to the square of Eq.(A.2) and to Eq. (A.3). Eq. (A.2) is important because (due to $\dot{\vartheta} > 0$ in $\mathscr{D}$) it implies

$$\mathrm{sign}\dot{\rho} = \mathrm{sign}\sin(\vartheta - Y) \ ; \tag{A.5}$$

this information is essential to determine Y as a function of (ξ, ϑ). Eqs. (A.1)-(A.5) yield the expressions for P, E, Y in Eqs. (3.11)-(3.13); from here, one also proves the bijectivity of the map $(\xi, \vartheta) \mapsto (P, E, Y, \vartheta) \equiv (I, \vartheta)$ and the expression (3.16)-(3.19) for its inverse.

To conclude, let us comment on the equations (3.20) for the apocenter ρ_+, the pericenter ρ_- and the orbital period T_{orb} of the unperturbed Kepler motion along an ellipse of parameter P and eccentricity E. The expressions for $\rho_\pm$ are consequences of the polar equation for the ellipse; the expression for T_{orb} follows from Kepler's third law

$$T_{orb} = \frac{2\pi}{\sqrt{GM}}\ (\text{semi-major axis})^{3/2} = \frac{2\pi}{\sqrt{GM}}\left(\frac{\rho_+}{2} + \frac{\rho_-}{2}\right)^{3/2} \tag{A.6}$$

and from the previously mentioned formulas for $\rho_\pm$.

B Appendix. Numerical computation of the functions $\boldsymbol{a^i_{(0)}}$.

Let us consider a general periodic system (2.4), and suppose we have the related functions $s, \mathtt{J}, \mathtt{R}, \mathtt{K}$, for τ within a finite interval $[0, U)$. Hereafter we outline a simple, but effective scheme to construct by mixed, numerical and analytical, techniques a function $\tau \in [0, U) \mapsto a^i_{(0)}(\tau)$ fulfilling *up to small errors* the inequality (2.50), for any given $i \in \{1, ..., d\}$; this method is easily implemented on MATHEMATICA. The scheme consists of four items.
(i) Take on the torus $\mathbf{T}$ a grid of equally spaced angles

$$\vartheta_q := [\frac{2\pi q}{Q}] \ , \qquad (q = 1, 2, ..., Q) \ . \tag{B.1}$$

(ii) Take in $[0, U)$ a grid of equally spaced instants

$$\tau_n := \frac{U}{N} n \ , \qquad n = 0, 1, ..., N-1 \ . \tag{B.2}$$

(iii) For each n, compute numerically

$$a^i_{(0)\,n} := \max_{q=1,...,Q} \left| \Big(s(\mathtt{J}(\tau_n), \vartheta_q) - \mathtt{R}(\tau_n) s(I_0, \vartheta_0) - \mathtt{K}(\tau_n) \Big)^i \right| \ ; \tag{B.3}$$

then, for all $\vartheta \in \mathbf{T}$,

$$\left| \Big(s(\mathtt{J}(\tau_n), \vartheta) - \mathtt{R}(\tau) s(I_0, \vartheta_0) - \mathtt{K}(\tau_n) \Big)^i \right| \leqslant a^i_{(0)\,n} \tag{B.4}$$

up to an error neglected in the sequel, which is of order $O(1/Q^2)$ for $Q \to +\infty$.
(iv) Interpolate the sequence a^i_n ($n = 0, ..., N$), finding a smooth function $\tau \in [0, U) \mapsto a^i_{(0)}(\tau)$ such that

$$a^i_{(0)}(\tau_n) = a^i_{(0)\,n} \qquad \text{for} \quad n = 0, 1, ..., N-1. \tag{B.5}$$

More precisely, we define $a^i_{(0)}$ to be the Lagrange polynomial (restricted to $[0, U)$) such that $a^i_{(0)}(\tau_n) = a^i_{(0)\,n}$ for all n; thus

$$a^i_{(0)}(\tau) = \sum_{n=0}^{N-1} a^i_{(0)\,n} \Big(\prod_{\substack{m=0 \\ m \neq n}}^{N-1} \frac{\tau - \tau_m}{\tau_n - \tau_m} \Big) \qquad \text{for } \tau \in [0, U) \ , \tag{B.6}$$

with τ^i_n and $a^i_{(0)\,n}$ as in Eqs. (B.2) (B.3). By construction, the inequality (2.50)

$$\left| \Big(s(\mathtt{J}(\tau),\vartheta) - \mathtt{R}(\tau) s(I_0,0) - \mathtt{K}(\tau) \Big)^i \right| \leqslant a^i_{(0)}(\tau) \qquad \text{for all } \vartheta \in \mathbf{T}$$

is fulfilled, up to a small error, at all points $\tau = \tau_n$; we assume the same to happen everywhere in $[0, U)$. Of course, being C^∞, the functions (B.6) fulfill the regularity conditions for application of the basic Proposition 2.1.

The above scheme (i)...(iv) has been employed to construct the functions $a^i_{(0)}$ ($i = {\scriptscriptstyle P}, {\scriptscriptstyle E}, {\scriptscriptstyle Y}$) in the numerical examples of Section 5 on the motions of satellites. In these examples, $Q = 30$ and $N = 100$ (with such an N, the Lagrange interpolating polynomials are computed very quickly by MATHEMATICA). Admittedly, the approach followed does not account for the small errors mentioned in items (iii) (iv): throughout the paper these errors, and the ones produced by the approximate inversion of the matrix $1 - \varepsilon \partial\alpha/\partial r$, are the only ones for which an analytical estimate is not provided.

C Appendix. The functions a^i_j, b^i,..., e^i_{jk} for the satellite problem.

As usually, the indices i, j, k range in $\{{\scriptscriptstyle P}, {\scriptscriptstyle E}, {\scriptscriptstyle Y}\}$.

Finding a^i_j. As an example, let us illustrate the determination of $a^{\scriptscriptstyle P}_{\scriptscriptstyle Y}$. To this purpose, we use Eq. (4.11) with $i = {\scriptscriptstyle P}, j = {\scriptscriptstyle Y}$, $I = \mathtt{J}(\tau) = (P_0, E_0, \mathtt{J}^{\scriptscriptstyle Y}(\tau))$ and $\delta I = \delta J$, giving

$$\mathscr{S}^{\scriptscriptstyle P}_{\scriptscriptstyle Y}(\mathtt{J}(\tau), \delta J, \vartheta) = \int_0^1 dx \frac{\partial s^{\scriptscriptstyle P}}{\partial Y}(P_0 + x\delta J^{\scriptscriptstyle P}, E_0 + x\delta J^{\scriptscriptstyle E}, \mathtt{J}^{\scriptscriptstyle Y}(\tau) + x\delta J^{\scriptscriptstyle Y}, \vartheta) \tag{C.1}$$

$$= \int_0^1 dx \frac{E_0 + x\delta J^{\scriptscriptstyle E}}{P_0 + x\delta J^{\scriptscriptstyle P}} \, \Big(3\sin(\vartheta + \mathtt{J}^{\scriptscriptstyle Y}(\tau) + x\delta J^{\scriptscriptstyle Y}) - \sin(3\vartheta - \mathtt{J}^{\scriptscriptstyle Y}(\tau) - x\delta J^{\scriptscriptstyle Y}) \Big) \; ;$$

in the last passage, the derivative $\partial s^{\scriptscriptstyle P}/\partial Y$ has been computed from Eq. (4.5). Now we take the absolute value in the last equation, and use the elementary inequalities $| \int_0^1 dx ...| \leqslant \int_0^1 dx |...|$, $|E_0 + x\delta J^{\scriptscriptstyle E}| \leqslant E_0 + |\delta J^{\scriptscriptstyle E}|$, $1/|P_0 + x\delta J^{\scriptscriptstyle P}| \leqslant 1/(P_0 - |\delta J^{\scriptscriptstyle P}|)$, $|\sin(..)| \leqslant 1$. In this way we get

$$|\mathscr{S}^{\scriptscriptstyle P}_{\scriptscriptstyle Y}(\mathtt{J}(\tau), \delta J, \vartheta)| \leqslant 4 \frac{E_0 + |\delta J^{\scriptscriptstyle E}|}{P_0 - |\delta J^{\scriptscriptstyle P}|} = a^{\scriptscriptstyle P}_{\scriptscriptstyle Y}(r)\big|_{r^i = |\delta J^i|} \; , \tag{C.2}$$

where $a^{\scriptscriptstyle P}_{\scriptscriptstyle Y}(r) := 4(E_0 + \delta r^{\scriptscriptstyle E})/(P_0 - r^{\scriptscriptstyle P})$; this definition of $a^{\scriptscriptstyle P}_{\scriptscriptstyle Y}$ agrees with Eq. (4.20) (where we use the abbreviations $E_+ := E_0 + r^{\scriptscriptstyle E}$, $P_- := P_0 - r^{\scriptscriptstyle P}$).

Finding b^i and c^i. We illustrate the determination of $b^{\scriptscriptstyle Y}$. To this purpose, we must first compute the $\scriptscriptstyle P$ component of the function on the left-hand side of Eq. (2.27) (for $\mathtt{J} = (\mathtt{J}^i)$ and $\delta J = (\delta J^i)$ $(i = {\scriptscriptstyle P}, {\scriptscriptstyle E}, {\scriptscriptstyle Y})$. Recalling again that $\mathtt{J}^{\scriptscriptstyle P}(\tau) = P_0$ and $\mathtt{J}^{\scriptscriptstyle E}(\tau) = E_0$ for all τ, one finds

$$\Big(w(\mathtt{J}(\tau) + \delta J, \vartheta) - \frac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau))\, v(\mathtt{J}(\tau) + \delta J, \vartheta)\Big)^{\scriptscriptstyle Y} \tag{C.3}$$

$$= \frac{1}{512 {P_0}^3 (P_0 + \delta J^{\scriptscriptstyle P})^4 (E_0 + \delta J^{\scriptscriptstyle E})^2} \sum_{k=0,\dots,10;\ell=-6,\dots,6} P_{k\ell} \sin(k\vartheta + \ell(\mathtt{J}^{\scriptscriptstyle Y}(\tau) + \delta J^{\scriptscriptstyle Y}))\ ,$$

where the coefficients $P_{k\ell}$ are polynomials in $P_0, P_0 + \delta J^{\scriptscriptstyle P}, E_0 + \delta J^{\scriptscriptstyle E}$; for example,

$$P_{01} = 256(E_0 + \delta J^{\scriptscriptstyle E})\big(-49 P_0^3 + 13 P_0^3 (E_0 + \delta J^{\scriptscriptstyle E})^2 - 16 (P_0 + \delta J^{\scriptscriptstyle P})^3 (E_0 + \delta J^{\scriptscriptstyle E})^2\big)\ . \tag{C.4}$$

This implies

$$\Big|\Big(w(\mathtt{J}(\tau) + \delta J, \vartheta) - \frac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau))\, v(\mathtt{J}(\tau) + \delta J, \vartheta)\Big)^{\scriptscriptstyle Y}\Big| \tag{C.5}$$

$$\leqslant \frac{1}{512 (P_0 - |\delta J^{\scriptscriptstyle P}|)^3 (E_0 - |\delta J^{\scriptscriptstyle E}|)^2} \sum_{k=0,\dots,10;\ell=-6,\dots,6} |P_{k\ell}|\ ;$$

the absolute value of each coefficient $P_{k\ell}$ is easily bounded, using fairly rough estimates such as $|P_{01}| \leqslant 256(E_0 + |\delta J^{\scriptscriptstyle E}|)\big(49 P_0^3 + 13 P_0^3 (E_0 + |\delta J^{\scriptscriptstyle E}|)^2 + 16 (P_0 + |\delta J^{\scriptscriptstyle P}|)^3 (E_0 + |\delta J^{\scriptscriptstyle E}|)^2\big)$. The conclusion is

$$\Big|w(\mathtt{J}(\tau) + \delta J, \vartheta) - \frac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau))\, v(\mathtt{J}(\tau) + \delta J, \vartheta)\Big|^{\scriptscriptstyle Y} \leqslant b^{\scriptscriptstyle Y}(r)\big|_{r^i = |\delta J^i|}\ , \tag{C.6}$$

where $b^{\scriptscriptstyle Y}$ is the function appearing in Eq. (4.21) (which is independent of $r^{\scriptscriptstyle Y}$). The determination of $b^{\scriptscriptstyle P}, b^{\scriptscriptstyle E}$ and $c^{\scriptscriptstyle P}, c^{\scriptscriptstyle E}, c^{\scriptscriptstyle Y}$ is performed similarly, yielding Eqs. (4.21) (4.22).

Finding d^i_j. As an example, let us consider the determination of $d^{\scriptscriptstyle P}_{\scriptscriptstyle Y}$. We use Eq. (4.12) with $i = {\scriptscriptstyle P}$, $j = {\scriptscriptstyle Y}$, $I = \mathtt{J}(\tau) = (P_0, E_0, \mathtt{J}^{\scriptscriptstyle Y}(\tau))$ and $\delta I = \delta J$, giving

$$\mathscr{G}^{\scriptscriptstyle P}_{\scriptscriptstyle Y}(\mathtt{J}(\tau), \delta J) = \int_0^1 dx \frac{\partial \overline{p}^{\,\scriptscriptstyle P}}{\partial Y}(E_0 + x\delta J^{\scriptscriptstyle E}, P_0 + x\delta J^{\scriptscriptstyle P}, \mathtt{J}^{\scriptscriptstyle Y}(\tau) + x\delta J^{\scriptscriptstyle Y}) \tag{C.7}$$

$$= -3\pi \int_0^1 dx \frac{(E_0 + x\delta J^{\scriptscriptstyle E})^2}{(P_0 + x\delta J^{\scriptscriptstyle P})^3} \cos(2\mathtt{J}^{\scriptscriptstyle Y}(\tau) + 2x\delta J^{\scriptscriptstyle Y})\ ;$$

in the last passage, the derivative $\partial \overline{p}^{\,\scriptscriptstyle P}/\partial Y$ has been computed from Eq. (4.8). The last equation implies

$$|\mathscr{G}^{\scriptscriptstyle P}_{\scriptscriptstyle Y}(\mathtt{J}(\tau), \delta J)| \leqslant 3\pi \frac{(E_0 + |\delta J^{\scriptscriptstyle E}|)^2}{(P_0 - |\delta J^{\scriptscriptstyle P}|)^3} = d^{\scriptscriptstyle P}_{\scriptscriptstyle Y}(r)\big|_{r^i = |\delta J^i|}\ , \tag{C.8}$$

with d^{P}_{Y} as in Eq. (4.23).

Finding e^i_{jk}. These functions should bind the absolute values of the components $\mathscr{H}^i_{jk}(\mathtt{J}(\tau), \delta J)$ of $\mathscr{H}$; these are provided by Eq. (4.13), with $I = \mathtt{J}(\tau)$ and $\delta I = \delta J$. From the cited equation, we see that we can take $e^i_{jk}(\tau, r) := 0$ for $(i, j, k) \neq (Y, P, P)$. Again from (4.13), we infer

$$|\mathscr{H}^{Y}_{PP}(\mathtt{J}(\tau), \delta J)| = 36\pi \left| \int_0^1 dx \frac{(1-x)}{(P_0 + x\delta J^{P})^4} \right| \tag{C.9}$$

$$\leqslant 36\pi \frac{\int_0^1 dx(1-x)}{(P_0 - |\delta J^{P}|)^4} = \frac{18\pi}{(P_0 - |\delta J^{P}|)^4} = e^{Y}_{PP}(r)\Big|_{r^i = |\delta J^i|},$$

with e^{Y}_{PP} as in Eq. (4.24) .

D Appendix. The approximate inverse (4.28).

We must invert the 3×3 matrix $1 - \varepsilon(\partial\alpha/\partial r)(r)$, whose elements are given by Eq. (4.27). For $(\varepsilon, r) \to 0$, this equation implies

$$1 - \varepsilon \frac{\partial \alpha}{\partial r}(r) = 1 - \varepsilon M - \varepsilon r^k N_{(k)} - \varepsilon^2 Z + O_3(\varepsilon, r) \qquad \text{for } (\varepsilon, r) \to 0 \ , \tag{D.1}$$

where we provisionally write $M, N_{(k)}$ and Z for the matrices of elements

$$M^i_j := a^i_j(0) \ , \quad {N_{(k)}}^i_j := \frac{\partial a^i_k}{\partial r^j}(0) + \frac{\partial a^i_j}{\partial r^k}(0) \ , \quad Z^i_j := \frac{\partial b^i}{\partial r^j}(0) \ , \tag{D.2}$$

$(i, j, k \in \{P, E, Y\})$. Eq. (D.1), with the standard $X \to 0$ expansion $(1 - X)^{-1} = 1 + X + X^2 + O_3(X)$, implies $(1 - \varepsilon(\partial\alpha/\partial r)(r))^{-1} = 1 + (\varepsilon M + \varepsilon r^k N_{(k)} + \varepsilon^2 Z)$ $+(\varepsilon M + \varepsilon r^k N_{(k)} + \varepsilon^2 Z)^2 + O_3(\varepsilon, r) = 1 + \varepsilon M + \varepsilon r^k N_{(k)} + \varepsilon^2 Z + \varepsilon^2 M^2 + O_3(\varepsilon, r)$. Summing up,

$$\left(1 - \varepsilon \frac{\partial \alpha}{\partial r}(r)\right)^{-1} = 1 + \varepsilon M + \varepsilon r^k N_{(k)} + \varepsilon^2 Q + O_3(\varepsilon, r) \ , \qquad Q := Z + M^2 \ . \tag{D.3}$$

This justifies the approximate inverse (4.28) if we check that the matrices $M, N_{(k)}$ and Q, defined here via Eq.s (D.2) (D.3), coincide with the homonymous matrices in Eq.s (4.29) (4.30) (4.31). This is obtained substituting in (D.2) (D.3) the explicit expressions (4.20) (4.21) of the functions a^i_j and b^i.

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